16.4: Green's Theorem

Green's Theorem states: On a positively oriented, simple closed curve C that encloses the region D where P and Q have continuous partial derivatives, we have

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

As noted in class, when working with positively oriented closed curve, C, we typically use the notation:

$$\oint_C P\,dx + Q\,dy = \int_C P\,dx + Q\,dy.$$

NOTES:

1. This theorem is for **closed curves**.

2. It is true for conservative and nonconservative vector fields. But for conservative vector fields the value of such an integral is just zero (remember that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ for a conservative vector field). So really this theorem is for nonconservative vector fields over closed curves.

3. This theorem gives an important relationship between the boundary a line integral of the boundary of a region and the double integral itself. These facts are useful in several ways:

   (a) *Computing a line integral faster*: This gives us options. If it is a pain to parameterize the closed curve, then we can instead do a double integral. Both ways work, but this theorem gives us options to choose a faster computation method.

   (b) *Computing a double integral with a line integral*: Sometimes it may be easier to work over the boundary than the interior. Green's theorem gives us a connection between the two so that we can compute over the boundary. For example we found that we can find the area of a two-dimensional region in several way using line integrals as follows:

$$\text{Area of } D = \iint_D 1dA = \oint_C -y\,dx = \oint_C x\,dy = \frac{1}{2}\oint_C -ydx + xdy$$

4. We will interpret the physical significance of this result more in subsequent chapters. For now you need to be able to compute with it. The following page contains two examples.

- Compute $\oint_C -2y^3\,dx + 2x^3\,dy$ where C is the circle of radius 3 centered at the origin.

  ANSWER: Using Green's theorem we need to describe the interior of the region in order to set up the bounds for our double integral. This is best described with polar coordinates, $0 \le \theta \le 2\pi$ and $0 \le r \le 3$. And we get

$$\begin{aligned}
\oint_C -2y^3\,dx + 2x^3\,dy &= \iint_D (6x^2 + 6y^2)dA \\
&= 6\int_0^{2\pi}\int_0^3 r^2 r\,dr\,d\theta \\
&= 6\int_0^{2\pi} \frac{1}{4}r^4\Big|_0^3 \,d\theta \\
&= 6\int_0^{2\pi} \frac{81}{4} d\theta \\
&= \frac{243}{2}\theta\Big|_0^{2\pi} \,dx = 243\pi
\end{aligned}$$

  So if $\mathbf{F}(x,y) = \langle -2y^3, 2x^3 \rangle$ was a force field say in units Newtons, then we just calculated WORK $= \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C -2y^3\,dx + 2x^3\,dy = 243\pi$ Joules.

- Compute $\oint_C x\,dx + xy^2\,dy$ where C is the triangle with vertices (0,0), (2,0), (2,6).

  ANSWER: Using Green's theorem we need to describe the interior of the region in order to set up the bounds for our double integral. The triangle has sides with equations (in x and y) of $y = 0$, $x = 2$ and $y = 3x$. If you graph the region, you see that it can be described as a 'top/bottom' region using $0 \le x \le 2$ with $0 \le y \le 3x$. And we get

$$\begin{aligned}
\oint_C x\,dx + xy^2\,dy &= \iint_D (y^2 - 0)dA \\
&= \int_0^2\int_0^{3x} y^2\,dy\,dx \\
&= \int_0^2 \frac{1}{3}y^3\Big|_0^{3x} \,dx \\
&= \int_0^2 9x^3 dx \\
&= \frac{9}{4}x^4\Big|_0^2 \,dx = 36
\end{aligned}$$

  Remember if $\mathbf{F}(x,y) = \langle x, xy^2 \rangle$ was a force field say in units Newtons, then we just calculated WORK $= \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C x\,dx + xy^2\,dy = 36$ Joules.