MATH 308 O

Final Exam

June 8, 2020

Name ________________________________

Student ID #____________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:________________________________

1	16	
2	8	
3	18	
4	12	
5	16	
Bonus	5	
Total	70	

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 110 minutes to complete the exam and there are 5 problems. Try not to spend more than 20 minutes on each problem. You will have 10 minutes at the end of the exam to upload your solutions to Gradescope.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- This is an open book exam, however, you are not allowed to collaborate with anyone.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. **Construct examples.** If you are asked to provide an example and there is no such example, write NOT POSSIBLE. No justification required.

   (a) (2 points) **Give an example** of a matrix A that represents the following transformation.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ y \end{bmatrix}.$$

   (b) (2 points) **Give an example** of a set of linearly dependent vectors in $\mathbb{R}^3$ such that when you remove **any one** of the vectors, the remaining set is linearly independent and spans $\mathbb{R}^3$.

   (c) (2 points) **Give an example** of a 3×3 matrix A with eigenvalues 1 and -4, where $\text{rank}(A) = 2$.

Short Answer Questions.

(d) (4 points) Let $\vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Find a basis $\mathscr{B}$ such that $[\vec{x}]_{\mathscr{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. **Reminder:** A basis is a *set of vectors*, not a matrix.

(e) (6 points) **Fill in the blanks.** Assume $p(\lambda) = (\lambda)(\lambda+2)^2(\lambda-2)(\lambda+4)^3$ is the characteristic polynomial of a matrix A. Then

i. A is a __________ $\times$ __________ matrix.

ii. The eigenvalues of A are __________________________

iii. Is A invertible? **Justify your answer.**

iv. Is A guaranteed to be diagonalizable? If so, justify your answer. If not, explain what you would need to know to guarantee A is diagonalizable.

2. (8 Points) Let A and B be $n \times n$ matrices, and determine if the following sets are subspaces of $\mathbb{R}^n$.

   (a) $S = \{\vec{v} \in \mathbb{R}^n : A^2\vec{v} = AB\vec{v}\}$

   (b) $S = \{\vec{v} \in \mathbb{R}^n : A^2\vec{v} - \vec{e}_1 = \vec{0}\}$

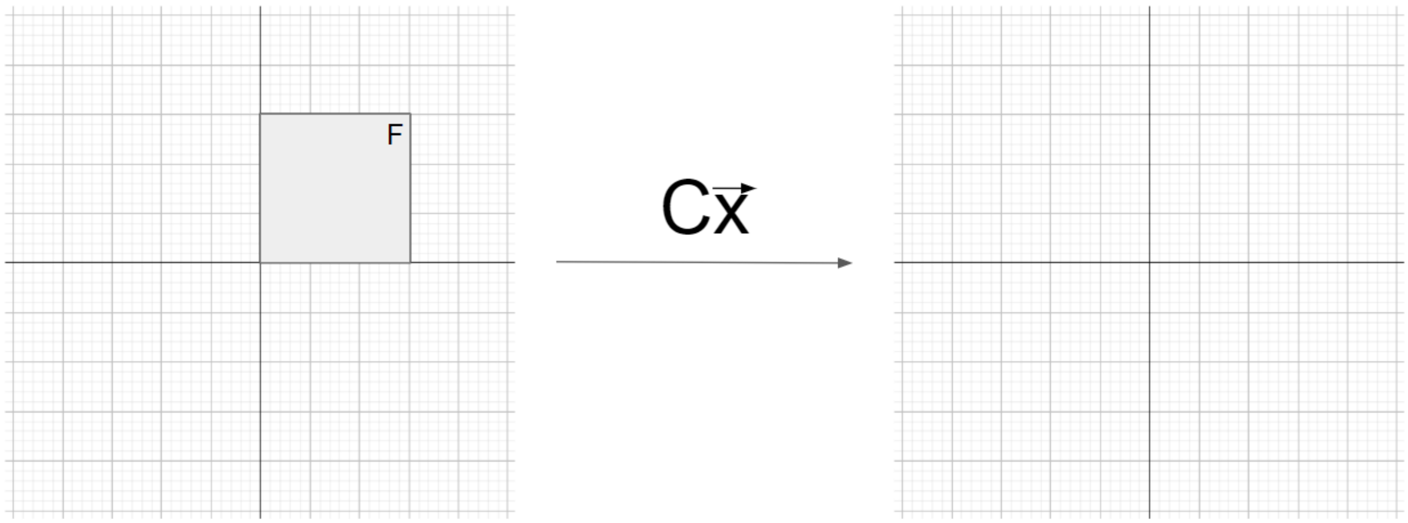
3. (a) (2 points) Produce a 2×2 matrix that reflects $\mathbb{R}^2$ over the y-axis. Call this matrix S.

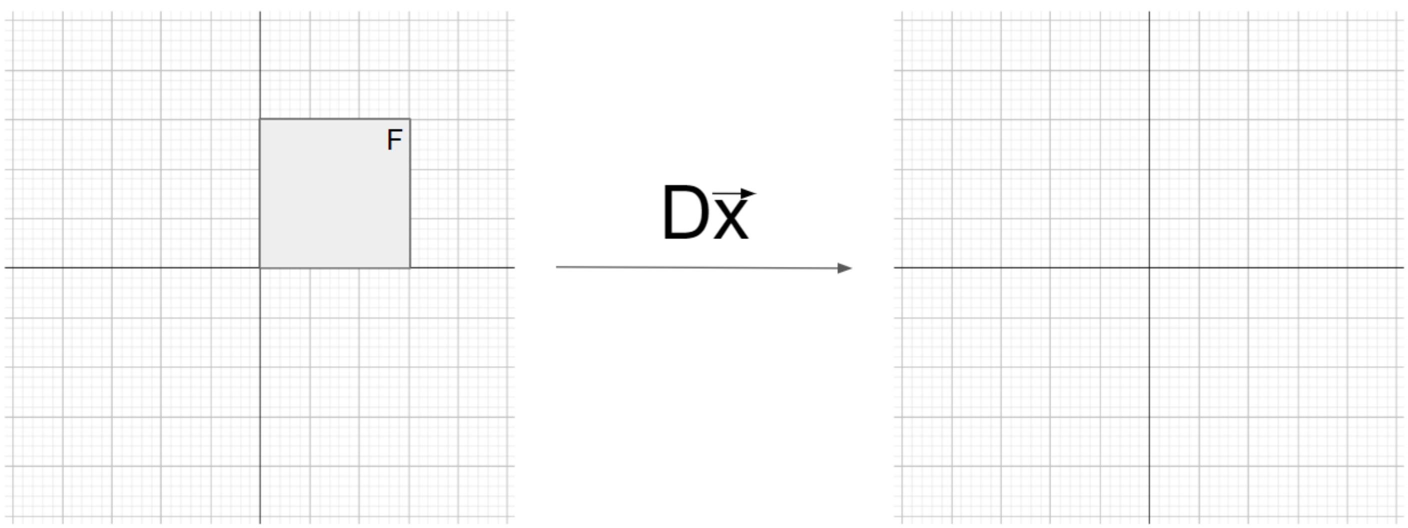
(b) (2 points) Produce a 2×2 matrix that rotates $\mathbb{R}^2$ by 90 degrees ($\frac{\pi}{2}$ radians) counter-clockwise. Call this matrix R.

(c) (2 points) Compute the matrix that represents a reflection of $\mathbb{R}^2$ over the y-axis then a rotation by 90 degrees counter-clockwise, **in that order.** Call this matrix C.

(d) (2 points) Compute the matrix that represents a rotation of $\mathbb{R}^2$ by 90 degrees **clockwise**, then a reflection over the y-axis, **in that order.** Call this matrix D.

(e) (4 points) Complete the following drawings. Show where the unit square gets mapped and draw F with the correct orientation on the new square. Matrix C denotes the matrix from part (c), and matrix D denotes the matrix from part (d).





(f) (1 point) What relationship do C and D have? What does that mean about S and R? Express the relationship between S and R.

(g) (1 point) What happens if you apply the matrix S twice? Use geometric intuition first and write out what you think will happen, then compute S^2 to justify.

(h) (4 points) Use part (f) and part (g) to simplify the following expression as much as possible: $SRSRSRSRSRSRS$.

4. Let A be a 3×3 matrix that satisfies the equation

$$A^3 + 2A^2 - I = 0$$

   (a) (4 points) Show that the matrices A and $A + 2I$ are invertible.

   (b) (4 points) If $\det(A) = \sqrt{3}$, what is the $\det(A + 2I)$?

   (c) (4 points) Explain why -2 is not an eigenvalue of A.

5. A linear transformation $T :\to \mathbb{R}^4 \to \mathbb{R}^3$ has the following properties:

- The vector $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is in the range(T), but $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is not.
- The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ both satisfy $T(\vec{v}_1) = T(\vec{v}_2) = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Answer the following questions about T. **Justify your answers.** And **do not** try to find the matrix representing T.

(a) (1 point) Is T one-to-one?

(b) (1 point) Is T onto?

(c) (2 points) Determine the dim(range(T)).

(d) (2 points) Find a basis for range(T).

(e) (3 points) Find a nonzero vector $\vec{x}$ such that $T(\vec{x}) = \vec{0}$.

(f) (3 points) Is your answer from part (e) a basis for ker(T)?

(g) (4 points) Find another vector $\vec{w}$ that is not $\vec{v}_1$ or $\vec{v}_2$ such that $T(\vec{w}) = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

BONUS: The transpose operation is a linear transformation on $n \times n$ matrices. By "linear transformation on $n \times n$ matrices" we mean that the transpose operation satisfies the two usual conditions in the definition, but instead of applying the transformation to a vector, we can apply it to a matrix.

(a) (1 point) Show that the transpose operation is a linear transformation on $n \times n$ matrices.

(b) (4 points) Since the transpose is a linear transformation, we can find a matrix that represents it. However, that requires understanding that a collection of $m \times m$ matrices can be thought of as some $\mathbb{R}^n$. Find a matrix that represents this linear transformation on 2×2 matrices. **Hint:** Can you write the matrix as a vector somehow? You must choose a basis. (No credit will be awarded if it is not clear what basis you have chosen.)