Lecture #8

In the last lecture, we finally landed on the big structural theorem of the course: the Howe-Moore vanishing theorem.

<u>THM</u> (Howe-Moore Vanishing Theorem)

Let $\pi: SL_2\mathbb{R} \longrightarrow \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation of $SL_2\mathbb{R}$ on a Hilbert space $\mathcal{H}$. Assume that the representation has <u>no</u> invariant vectors (i.e. no $v \in \mathcal{H}$ such that for any $g \in G$, $\pi(g)v = v$). Then all matrix coefficients of π vanish at ∞.

The interested reader can find the more general form of the Howe-Moore theorem in Bekka-Mayer (see §3, Theorem 1.1). We refrain from stating it here because we would need a bit more Lie algebra theory to make sense of it.

What is magical about the Howe-Moore theorem is that it is a result that shows us that the topological and algebraic structure of $SL_2\mathbb{R}$ (and more generally, semisimple Lie groups) governs its dynamical properties in a rigid way.

For instance, let $SL_2\mathbb{R} \curvearrowright (X,\mu)$ be an action by measure-preserving transformations on a probability space. The Howe-Moore theorem tells us that if the action is ergodic, then it is <u>mixing</u>! Indeed, if the action is ergodic, then the only invariant vectors in $L^2(X,\mu)$ are the constant functions (with respect to the associated unitary representation π). That means that $L^2_0(X,\mu)$ has no invariant vectors, and by Howe-Moore, all matrix coefficients of π vanish. Thus, $SL_2\mathbb{R} \curvearrowright (X,\mu)$ is <u>mixing</u>.

More is true. Let H be any non-compact subgroup of $SL_2\mathbb{R}$, for example, $H = A$, the diagonal subgroup. Then if $SL_2\mathbb{R} \curvearrowright (X,\mu)$ is ergodic, not only is it mixing, but so is the restriction of the action to H. This is the principal behind Moore's ergodicity theorem, which like Howe-Moore, has a more general form applicable to semisimple Lie groups. See Bekka-Mayer (§3, Theorem 2.1 and Theorem 2.5).

Today, our goal is to apply Howe-Moore in a non-trivial way. We start with the following immediate application: Howe-Moore is a key step in observing that the geodesic flow is mixing on a finite-volume, hyperbolic manifold. Our focus will be on finite-volume, hyperbolic surfaces. Following, we will use the fact that the geodesic flow is mixing to count lattice points in the hyperbolic plane – the connection to the geodesic flow is bridged by an equidistribution result.

First, note that there is an action (by isometries!) of $SL_2\mathbb{R}$ on $\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper halfplane given by Möbius transformations.

For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, let

$$g \cdot z = \frac{az+b}{cz+d} \in \mathbb{H}^2.$$

In fact, note that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ act the same since

$$\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \cdot z = \frac{-az+(-b)}{-cz+(-d)} = \frac{(-1)(az+b)}{(-1)(cz+d)} = \frac{az+b}{cz+d} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z,$$

So, the group $PSL_2\mathbb{R} = SL_2\mathbb{R}/\{\pm Id\}$ is acting also.

The action by Möbius transformations is by orientation-preserving isometry, where the Riemannian metric on $\mathbb{H}^2$ is given by $ds^2 = \dfrac{dx^2+dy^2}{y^2}$,

where $z = x+iy$. (Note that with this metric, the norm on the tangent plane $T_z\mathbb{H}^2 \cong \mathbb{C}$ at $z \in \mathbb{H}$ is given by $\|v\|_z = \dfrac{|v|}{Im(z)}$.)

In fact, this action extends to an action on the unit tangent bundle of the upper half-plane, $T^1\mathbb{H}^2$, by

$$g\cdot(z,v) = (\underset{\substack{\uparrow\\ \text{Möbius}}}{g\cdot z},\ \underset{\substack{\uparrow\\ \text{differential of}\\ \text{Möbius transformation}\\ \text{at } z.}}{D_z g(v)})$$

(*)(See LEMMA + Exercise on the bottom of the next page)(*)

<u>EXERCISE</u> Show that for $g = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$, we have that for $v \in T_z\mathbb{H}^2$, $D_z g(v) = \dfrac{1}{(cz+d)^2}\cdot v$. Use this to verify that $PSL_2\mathbb{R} \curvearrowright \mathbb{H}^2$ is by isometries (Show $\|D_z g(v)\|_{g\cdot z} = \|v\|_z$, and conclude accordingly.)

Claim The geodesics (curves which are locally length minimizing) are the half-circles with (Euclidean) center on the real-axis and the half-lines parallel to the imaginary axis.

EXERCISE Show that $\gamma : [a,b] \longrightarrow \mathbb{H}^2$ given by $\gamma(a+t) = i(a+t)$ is length minimizing. Conclude that the imaginary axis is a geodesic in $\mathbb{H}^2$. Then, observe that the image of the imaginary axis in $\mathbb{H}^2$ under the action of $PSL_2\mathbb{R}$ is precisely the set described in the claim. Lastly, confirm that $PSL_2\mathbb{R}$ comprises <u>all</u> of the orientation-preserving isometries of $\mathbb{H}^2$. (Note that the full isometry group is generated by $PSL_2\mathbb{R}$ and the involution $z \longmapsto -\bar{z}$.) This proves the claim.

LEMMA $PSL_2\mathbb{R} \curvearrowright T^1\mathbb{H}^2$ by $g \cdot (z,v) = (g \cdot z, D_z g(v))$ is transitive and free.

EXERCISE Prove the Lemma.

NOTE We also have an action of $SL_2\mathbb{R}$ on $\mathbb{H}^2$ by isometries, but it is not free. Moreover, $\mathbb{H}^2$ has infinite volume, so the natural measure that one gets from the volume does not make $\mathbb{H}^2$ a probability space. However...

the following Proposition connects $SL_2\mathbb{R}$ to the geodesic flow...

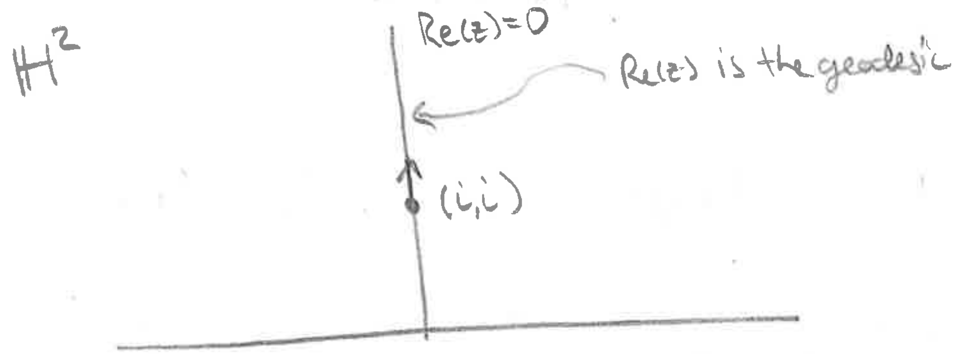
Prop The geodesic flow on $T^1\mathbb{H}^2$ corresponds to the flow on the group $PSL_2\mathbb{R}$ given by the right action

$$g \longmapsto g \cdot g_t$$

where $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ for all $t \in \mathbb{R}$.

NOTE Since g_t as above defines the one-parameter subgroup in $SL_2\mathbb{R}$, we may instead view this as a right action of $SL_2\mathbb{R}$ on $PSL_2\mathbb{R}$, restricted to g_t, and this action corresponds to the geodesic flow.

To see why this is the right way to view the geodesic flow on $T^1\mathbb{H}^2$ through the action of $PSL_2\mathbb{R}$ (or $SL_2\mathbb{R}$), consider the following. Fix $(i,i) \in T^1(\mathbb{H}^2)$. The corresponding geodesic runs along the imaginary axis: $(it, i) \in T^1\mathbb{H}$, $t \in \mathbb{R}$. (Really, forward time would be for $t \geq 0$ ... but we know the geodesic from our previous observations.)



If we apply a Möbius transformation to the imaginary axis, we move this geodesic to a new geodesic... but notice that if we apply g_t to $(i,i) \in T^1\mathbb{H}^2$, we get

$$\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \cdot (i,i) = \left(\frac{e^{t/2} i}{e^{-t/2}}, \frac{1}{(e^{-t/2})^2} \cdot i \right) = (ie^t, (e^t) \cdot i) \in T^1\mathbb{H}^2.$$

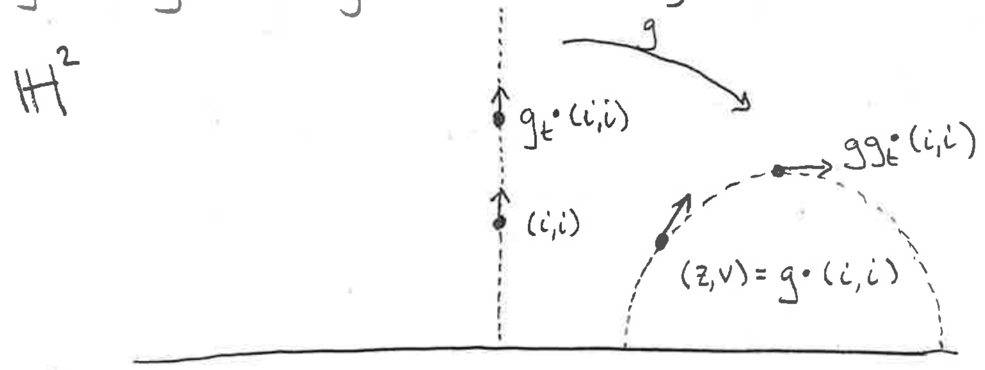
Indeed, $(ie^t, (e^t)\cdot i) \in T^1\mathbb{H}^2$:

$$\|e^t \cdot i\|_{ie^t} = \frac{|e^t \cdot i|}{\mathrm{Im}(ie^t)} = \frac{e^t}{e^t} = 1.$$

In other words, we "flow" along the imaginary axis in the "i" direction (or "north" direction) as t varies. In particular, for $A = \left\{ \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$,

$A \cdot (i,i) =$ Imaginary axis with a "north" vector.

Now, notice that to flow along a geodesic starting at some point $z \in \mathbb{H}$ in a direction $v \in T_z^1\mathbb{H}^2$, we first recognize that (z,v) is the image of (i,i), $(z,v) = g\cdot(i,i)$ for some $g \in PSL_2\mathbb{R}$ since the action is transitive. Then, $gg_t \cdot (i,i)$ is flowing along the geodesic starting at z in the direction v:



<u>NOTE</u> Observe that the action by $SO(2)$ (or $PSO(2)$) on the right will rotate the tangent vector at i. Indeed, we have $\mathrm{Stab}_{SL_2\mathbb{R}}(i) = SO(2)$ (or $\mathrm{Stab}_{PSL_2\mathbb{R}}(i) = PSO(2)$) and $\mathbb{H}^2 \cong SL_2\mathbb{R}/SO(2) \cong PSL_2\mathbb{R}/PS$

The last bit of structure that we need on $\mathbb{H}^2$ and $T^1\mathbb{H}^2$ is a measure. Let $d\mu := \frac{dx\,dy}{y^2}$ be the measure on $\mathbb{H}^2$.

<u>EXERCISE</u> Show μ is preserved by the action of $SL_2\mathbb{R}$.
(Recall: this action is an isometry!)

To get a measure m on $T^1\mathbb{H}^2$, we let dk denote the angle measure on S^1, observe that $T^1\mathbb{H}^2 \cong \mathbb{H}^2 \times S^1$, and define $dm := d\mu(z)\,dk$.

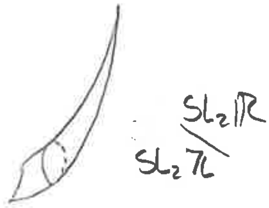
<u>EXERCISE</u> $PSL_2\mathbb{R} \curvearrowright (T^1\mathbb{H}^2, m)$ is measure-preserving.
Thus, identifying $PSL_2\mathbb{R}$ to $T^1\mathbb{H}^2$, then pushing the measure m to $PSL_2\mathbb{R}$, gives us a left Haar measure. (which is also a right Haar measure!).
i.e. $PSL_2\mathbb{R}$ is unimodular. <u>Note</u>: this implies lattices exist!

Now, we can reduce to a finite measure space by considering a quotient of $\mathbb{H}^2$ by a lattice $\Gamma \subset PSL_2\mathbb{R}$, which is a discrete subgroup of $PSL_2\mathbb{R}$ (with the quotient topology) such that $\Gamma\backslash\mathbb{H}^2$ is finite <u>volume</u>.

This construction allows for both compact hyperbolic surfaces

and non-compact, finite volume hyperbolic surfaces (surfaces with cusps).

"leaky torus"



Moreover, we have that

$$T^1(\Gamma\backslash\mathbb{H}^2) \cong \Gamma\backslash T^1\mathbb{H}^2,$$

i.e. we can identify the unit tangent bundle with $\Gamma\backslash PSL_2\mathbb{R}$,

and the right action by g_t gives us the geodesic flow just as before! In fact, we get a measure arising from the quotient measure of m, call it m_Γ.

We have:

<u>THM</u> The geodesic flow on $T^1(\Gamma\backslash\mathbb{H}^2)$, where $\Gamma\backslash\mathbb{H}^2$ has finite volume, is mixing.

pf: By the Hopf Argument, one can show that $\Gamma \backslash PSL_2\mathbb{R} \curvearrowleft A$ is ergodic, where $A = \{g_t \in SL_2\mathbb{R} \mid t \in \mathbb{R}\}$.

By Mautner's phenomenon, we have that N is ergodic. We showed this implied N^+ acts ergodically, and hence $\langle N \cup N^+ \rangle = SL_2\mathbb{R}$ acts ergodically (in the last Lemma of Lecture #7).

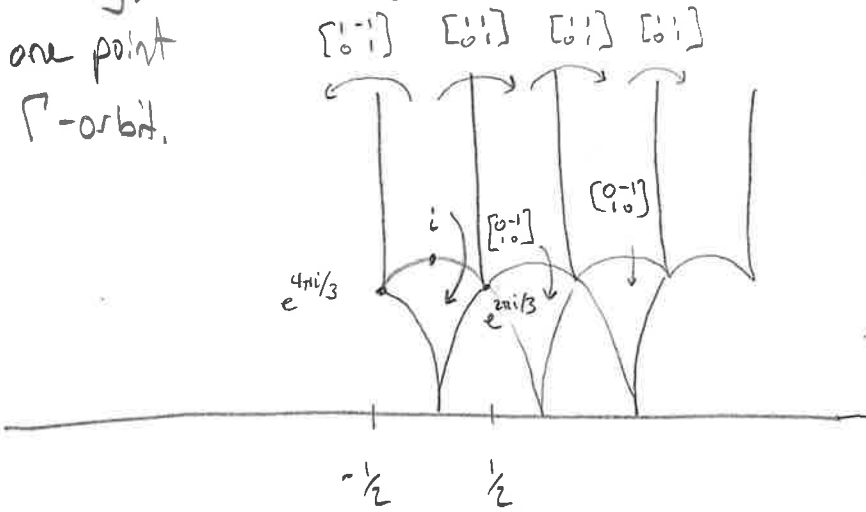
Thus, $\Gamma \backslash PSL_2\mathbb{R} \curvearrowleft SL_2\mathbb{R}$ is mixing, by Howe-Moore. Then, since A is a non-compact subgroup of $SL_2\mathbb{R}$, $\Gamma \backslash PSL_2\mathbb{R} \curvearrowleft A$ is mixing. ▨

Now, we can use the fact that the geodesic flow is mixing to prove an <u>equidistribution</u> result (saying measures along circles, when pushed by the geodesic flow, "equidistribute", i.e. converge to a measure which happens to be the Haar measure). This equidistribution result is interesting in its own right, but for us, is a step in a lattice point counting argument.

Here is a glimpse at the big picture. The following is the lattice point counting question we aim to answer.

<u>Q</u>: Fix a lattice $\Gamma \subset PSL_2\mathbb{R}$ (eg. $PSL_2\mathbb{Z}$). Fix a point $z \in \mathbb{H}^2$. What can be said about the number $N(R, q)$ of points in the orbit $\Gamma \cdot z$ which lie in a ball of radius R about some point p, $B_R(p)$?

Intuitively, there should be roughly "1" pt per lift of $\Gamma\backslash\mathbb{H}^2$ (under the quotient map $\pi_\Gamma : \mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$) that intersects $B_R(p)$. To see this, construct a fundamental domain for the action of Γ on $\mathbb{H}^2$. By definition (ignoring the boundary) each tiling of the fundamental domain (see below) contains one point in the Γ-orbit.



$PSL_2\mathbb{Z} = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle$

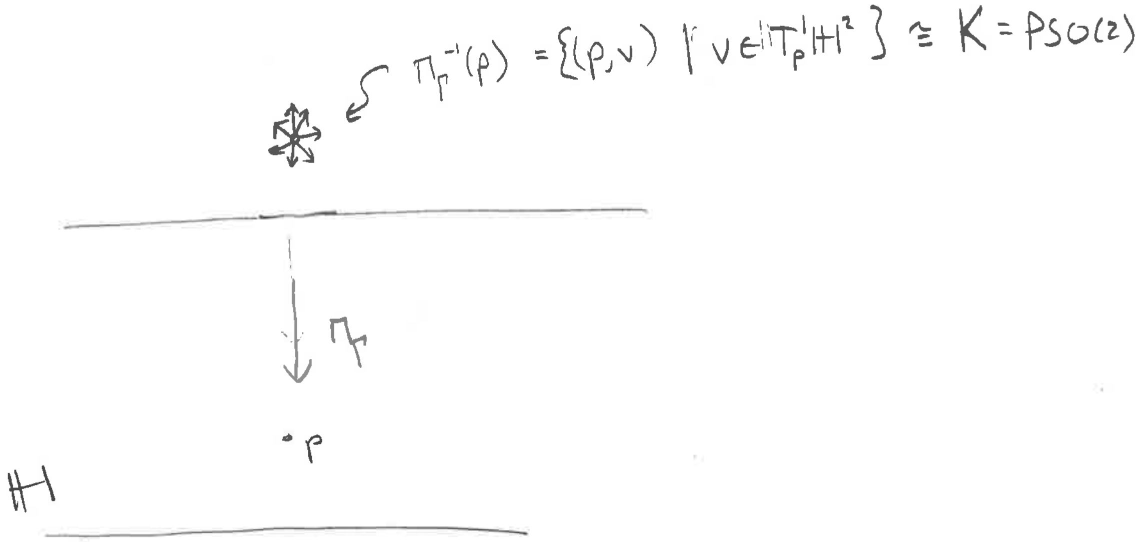
← Tiling $\mathbb{H}^2$ with a fundamental domain of $PSL_2\mathbb{Z}$.

In other words, one should expect $N(R, q) \sim \dfrac{\text{Area}(B_R(p))}{\text{Area}(\Gamma\backslash\mathbb{H}^2)}$

Returning to the equidistribution result, here is the set-up:

Let Γ be torsion-free ($\Gamma \cap PSO(2) = \{e\}$). Let

$p \in \mathbb{H}$, and consider its preimage in $T^1\mathbb{H}^2$:



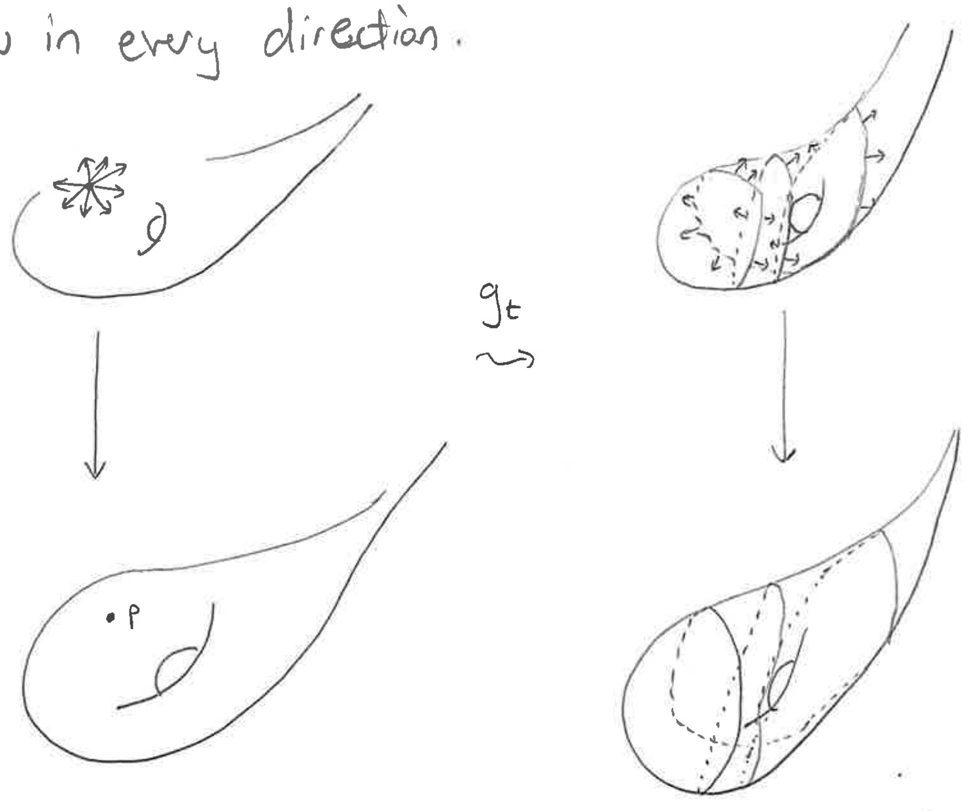
If we apply the geodesic flow to $\pi_\Gamma^{-1}(p)$, we flow along in every direction. After time t, if we project, we see a "circle". Moreover, WLOG, we can assume $p = i$ (since the isometry group is transitive), so, $\pi_\Gamma^{-1}(p) = K = PSO(2)$.

Observe that if we want to integrate a function $f \in C(\mathbb{H}^2)$ just over the circle, we integrate the "line integral":

$$\frac{1}{m_K(K)} \int_{K=PSO(2)} f(k g_t \cdot p)\, dk .$$

normalized so the measure of $PSO(2)$ is 1, if wanted. $\left(m_K(K) = \int_K dk \right)$

We have the same set-up on a hyperbolic surface, for Γ torsion-free (to simplify the argument). Pick a point $p \in \Gamma\backslash\mathbb{H}^2$, lift this to $T^1(\Gamma\backslash\mathbb{H}^2)$, and apply the geodesic flow in every direction.



Question If we take a function $f \in C_c(\Gamma\backslash\mathbb{H})$, what happens to the circle averages (here, $[kg_t \cdot p]$ is the equivalence class!):

$$\frac{1}{m_K(K)} \int_{K = PSO(2)} f([kg_t \cdot p])\, dk$$

normalized so the measure of $SO(2)$ is 1, if wanted.

$$\left(PSL_2\mathbb{R}/PSO(2) \cong SL_2\mathbb{R}/SO(2) \right)$$

as $t \to \infty$? The circle gets bigger and bigger ... does it cover the whole space? In what sense?

The answer is that under g_t, the circles equidistribute to the Haar measure, or rather, the normalized measures on the circles limit to the Haar measure (the natural measure on $T'(\Gamma \backslash \mathbb{H}^2)$ corresponding to the Haar measure on $\Gamma \backslash PSL_2\mathbb{R}$).

<u>THM</u> For all $f \in C_c(\Gamma \backslash \mathbb{H}^2)$, Γ torsion-free,

$$\frac{1}{m_K(K)} \int_K f([kg_t \cdot p])\, dk \xrightarrow{t \to \infty} \frac{1}{\text{area}(\Gamma \backslash \mathbb{H}^2)} \int_{\Gamma \backslash \mathbb{H}^2} f \, d\mu ;$$

<u>NOTE</u> μ is really the pushforward of μ in $\mathbb{H}^2$ under the quotient!

<u>EXERCISE</u> Rewrite this as equidistribution of measures λ_t clearly defining λ_t. ($\lambda_t \to m_\Gamma$ iff $\lambda_t(f) := \int_K f \, d\lambda_t \to m_\Gamma(f)$ for any compactly supported, continuous function f.)

The proof has two basic steps: first, we show that integrals over circles are close to integrals over "thickened" circles. Note that the Haar measure cannot see the circles: $\int_{\Gamma\backslash PSL_2\mathbb{R}} \chi_{\Gamma K}\, dm(g) = 0$. So, we need to slightly thicken the circle without changing the value of the integral much. Showing this will require two things: first, uniform continuity of the function f. This comes from the fact that f is compactly supported. Second, we need to know that our method of thickening the circle is controlled in the sense that as we push the thickened circles by the geodesic flow, integrals over the geodesic push of the thickened circles (which are much larger circles) are still comparable to integrals over the geodesic push of non-thickened circles. (There is a commutativity issue hiding!) This is solved with a "toy" version of something called the Wavefront Lemma.

The second step uses mixing of the geodesic flow: we recognize the thickened integrals as matrix coefficients and apply mixing.

pf: Let $f \in C_c(\Gamma \backslash \mathbb{H}^2)$ and identify it with a function $f \in C_c(\Gamma \backslash SL_2\mathbb{R} / SO(2))$ by identifying $\Gamma \backslash \mathbb{H}^2$ with $\Gamma \backslash SL_2\mathbb{R}/SO(2)$. (We are assuming $\Gamma \subset \mathrm{Isom}(\mathbb{H}^+)$ is being identified with $\Gamma \subset SL_2\mathbb{R}$.) It will be easier to lift everything into $\Gamma \backslash SL_2\mathbb{R} \cong T^1(\Gamma \backslash \mathbb{H}^2)$, so let $\tilde{f} \in C_c(\Gamma \backslash SL_2\mathbb{R})$ be a lift of f such that

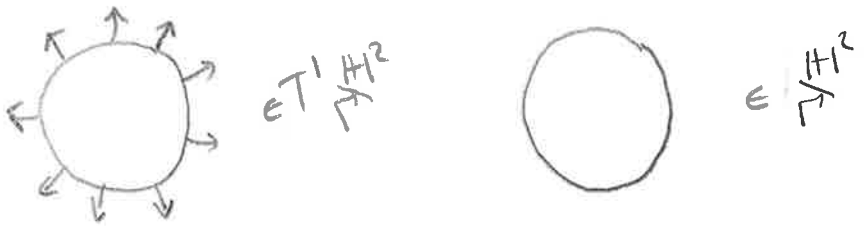
$$\tilde{f}(\Gamma g) = f(x)$$

when Γg is identified to $(x,v) \in T^1(\Gamma \backslash \mathbb{H}^2)$.

Observe that, for $m_K(K) = \int_K dk$,

$$\frac{1}{m_K(K)} \int_K \tilde{f}(\Gamma k g_t)\, dk = \frac{1}{m_K(K)} \int_K f([k g_t \cdot p])\, dk,$$

where the left-hand side can be identified with an integral over a circle lifted to $T^1(\Gamma \backslash \mathbb{H}^2)$, but only taking the outward normal vectors:



Moreover, observe that

$$\frac{1}{\mu(\Gamma \backslash \mathbb{H}^2)} \int_{\Gamma \backslash \mathbb{H}^2} f([p])\, d\mu([p]) = \frac{1}{m_\Gamma(\Gamma \backslash SL_2\mathbb{R})} \int_{\Gamma \backslash SL_2\mathbb{R}} \tilde{f}(\Gamma g)\, dm_\Gamma$$

provided $m_K(K)=1$, where the m_K comes from the angle measure dk. Fix m_K in this way.

To see why this is possible, we need to understand how the Haar measure on $SL_2\mathbb{R}$ decomposes (keeping in mind that the measure m_Γ on $\Gamma \backslash SL_2\mathbb{R}$ is essentially the pushforward of the Haar measure m. For $\pi_\Gamma|_{\mathcal{F}} : \mathcal{F} \longrightarrow \Gamma\backslash SL_2\mathbb{R}$, where $\mathcal{F}$ is a fundamental domain, $m_\Gamma = (\pi_\Gamma|_{\mathcal{F}})_* m$.)

First, we can decompose $SL_2\mathbb{R}$ into:

① KA^+K (Cartan, also a "Polar" decomposition)

or

② KAN (Iwasawa),

where ② is uniquely given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

where $\alpha = a^2+c^2$, $\cos\theta = \frac{a}{\sqrt{a^2+c^2}}$, $n = \frac{ab+cd}{\sqrt{a^2+c^2}}$.

We can use these decompositions to write the Haar measure (see Lang, $SL_2\mathbb{R}$):

$dm = dk\,da\,dk$, with appropriately normalization

$ = dk\,da\,dn$, with appropriate normalization.

(There is a slight lie here, see Lang - how we parametrize K, A, and N matters! Especially A...)

For the purposes of this proof, we will make the following observation: the left integral (taken with respect to $d\mu$) is an integral over $\Gamma \backslash SL_2\mathbb{R}/SO(2)$, so a "K" has been quotiented out. The right-hand side puts the K back in, and the function takes the same values over this K in the lift.

Caution There are 2 K's at play! The K generating the circle that will eventually equidistribute, and another K that plays the role of the unit tangent space at each point.

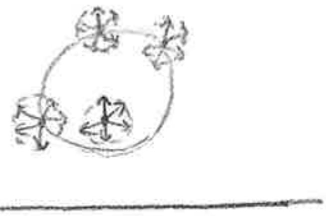
$\mathbb{H}^2$

$\mathbb{H}^2$ foliated by circles, ie.

$\mathbb{H}^2 \cong KA^+$:

$\rightsquigarrow$

$T^1\mathbb{H}^2$



$T^1\mathbb{H}^2 \cong KA^+K$

In Eskin-McMullen's work, the first K is called $\underset{\sim}{H}$, because in the general setting, the subgroup H ("circles") which equidistribute differ from K (maximal compact). They coincide here!

Now, back to the integrals. To "thicken" the integral

$$\frac{1}{m_K(K)} \int_K \tilde{f}(\Gamma \cdot k g_t) dk$$

we need to thicken the domain of integration to some $K \cdot V$, for some small $V \ni e$. (Notice, KV will be decomposable, so this amounts to picking $V = K \cdot V_a V_n$, $V_a \subset A$, $V_n \subset N$ or $V = K V_a V_k$, $V_a \subset A^+$; $V_k \subset K$. For dynamical reasons, the former will be a better choice!) However, we also need to ensure that the integrand won't change much.

So, to control this, fix $\varepsilon > 0$. Since f is uniformly continuous, there exists $U^{open} \ni e$ such that

$$\left| \tilde{f}(\Gamma g u) - \tilde{f}(\Gamma g) \right| < \varepsilon \qquad (*)$$

for any $g \in SL_2\mathbb{R}$, any $u \in U^{open}$.

Now, we thicken in a way such that for every element in the thickened set, $(*)$ is true.

Here is the issue. Let V be a small open neighborhood of the identity. The thickened circle is $K\tilde{V} = K.V$ ($\tilde{V} \subset AN$) and we can integrate

like "dadn", but possibly with scaling.

$$\frac{1}{m_K(K)} \frac{1}{m_{\tilde{V}}(\tilde{V})} \int_{\tilde{V}} \int_K \tilde{f}(\Gamma \cdot k v g_t) \, dk \, d\tilde{v} \ldots$$

but notice that

$$|\tilde{f}(\Gamma k v g_t) - \tilde{f}(\Gamma k g_t)|$$

is not controlled, even if $v \in U$. We are now multiplying on the wrong side! We must pick V carefully; and we need some sort of commuting property with A to make this work.

LEMMA (Toy version of the Wavefront Lemma)

For any neighborhood $U \ni e$ in $SL_2\mathbb{R}$, there exists a neighborhood V of the identity in $SL_2\mathbb{R}$ such that for all $g \in AK$,

$$KVg \subset KgU.$$

(Take a moment to compare this to the issue raised above. Why does it resolve the issue?)

pf of Lemma: Fix $U \ni e$ in $SL_2\mathbb{R}$.

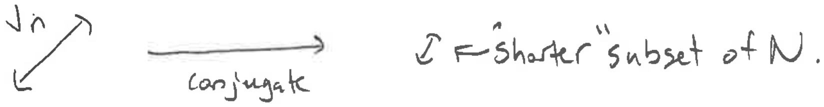
Case 1: Assume $g = g_t \in A^+$ first, where $A^+ = \left\{ \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \mid t \geq 0 \right\}$.

Recall that for any $h \in N = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{R} \right\}$, we have that $g^{-m} h g^{m} \xrightarrow{m \to \infty} e$. (LEMMA from Lecture 6). Moreover, $\forall\, t \geq 0$,

$$\begin{bmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{t/2} \end{bmatrix} = \begin{bmatrix} 1 & ne^{-t} \\ 0 & 1 \end{bmatrix},$$

So we can pick $V_n^{\ni e} \subset N$ such that

$$g_t^{-1} V_n g_t \subset V_n \quad \forall\, t \geq 0 \qquad (**)$$



Thus, by picking $V_a \subset A$ such that $\boxed{V_a V_n \subset U}$

(shrinking V_n further if necessary):

$\left(\underbrace{V_a V_n \subset AN}_{\text{So a "slice" in } U\ldots} \right)$

Observe that since A is abelian, we have

$$g_t^{-1} V_a V_n g_t = V_a g_t^{-1} V_n g_t \subset V_a V_n,$$

i.e.

$$V_a V_n g_t \subseteq g_t V_a V_n \qquad \forall\, t \geq 0.$$

Let, $V = K V_a V_n$, Then

$$\begin{aligned} K V g_t = K K V_a V_n g_t = K V_a V_n g_t &= K V_a g_t g_t^{-1} V_n g_t \\ &= K g_t V_a g_t^{-1} V_n g_t \\ &\subseteq K g_t V_a V_n \\ &\subseteq K g_t U, \end{aligned}$$

which completes case 1.

Case 2: Assume $g = g_t \in A^- = \left\{ \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \mid t \leq 0 \right\}$.

Exercise: construct V so that the Lemma is satisfied.

Case 3: Let $g \in AK$. Since K is compact, we can find $\mathcal{U}' \subset \mathcal{U}$ such that $k^{-1}\mathcal{U}'k \subset \mathcal{U}$ for all $k \in K$.

Choose V so that $HVa \subset Ha\mathcal{U}'$ for all $a \in A$

(by running Case 1, generating V_1, then case 2, generating V_2, and taking $V = V_1 \cap V_2$).

Then, $\forall\, g \in AK$

$$HVg = HVak \subset Ha\mathcal{U}'k = Hakk^{-1}\mathcal{U}'k$$
$$\subset Hak\mathcal{U}$$
$$= Hg\mathcal{U}.$$

$\boxtimes$

With this, we can now return to the proof of circle equidistribution and complete the first step. Pick V satisfying the Wavefront Lemma with respect to $\mathcal{U}$ chosen prior.

Let $\tilde{V} \subset AN$ be such that $K\tilde{V} = KV$. (Restrict if necessary so that $m(V) = m_\Gamma(\Gamma V)$).

$$\frac{1}{m_K(K)} \cdot \frac{1}{m_{\tilde{V}}(\tilde{V})} \int_{\tilde{V}} \int_K \tilde{f}(\Gamma k v g_t)\, dk\, d\tilde{v}$$

$$= \frac{1}{m_K(K)} \cdot \frac{1}{m_{\tilde{V}}(\tilde{V})} \int_{\tilde{V}} \int_K \tilde{f}(\Gamma k g_t u_v)\, dk\, d\tilde{v}$$

for some $u_v \in \mathcal{U}$, depending on v.

So, we have

$$\left| \underbrace{\frac{1}{m_K(K)} \int_K \tilde{f}(\Gamma k g_t)\, dk} - \frac{1}{m_K(K)} \cdot \frac{1}{m_{\tilde{V}}(\tilde{V})} \int_{\tilde{V}} \int_K \tilde{f}(\Gamma k g_t u_v)\, dk\, d\tilde{v} \right.$$

$$= \left| \frac{1}{m_K(K)} \cdot \frac{1}{m_{\tilde{V}}(\tilde{V})} \int_{\tilde{V}} d\tilde{v} \cdot \int_K \tilde{f}(\Gamma k g_t)\, dk \right.$$

$$\left. - \frac{1}{m_K(K)} \cdot \frac{1}{m_{\tilde{V}}(\tilde{V})} \int_{\tilde{V}} \int_K \tilde{f}(\Gamma k g_t u_v)\, dk\, d\tilde{v} \right|$$

$\downarrow$ Fubini

$$= \left| \frac{1}{m_K(K)} \cdot \frac{1}{m_{\tilde{V}}(\tilde{V})} \int_{\tilde{V}} \int_K \tilde{f}(\Gamma k g_t)\, dk - \frac{1}{m_K(K)} \cdot \frac{1}{m_{\tilde{V}}(\tilde{V})} \int_{\tilde{V}} \int_K \tilde{f}(\Gamma k g_t u_v)\, dk\, d\tilde{v} \right|$$

$$\leq \frac{1}{m_K(K)} \cdot \frac{1}{m_{\tilde{V}}(\tilde{V})} \int_{\tilde{V}} \int_K \left| \underbrace{\tilde{f}(\Gamma k g_t) - \tilde{f}(\Gamma k g_t u_v)}_{< \varepsilon \text{ by uniform continuity!}} \right| dk\, d\tilde{v}$$

$$= \varepsilon \cdot \frac{1}{m_K(K)} \cdot \frac{1}{m_{\tilde{V}}(\tilde{V})} \cdot m_K(K) \cdot m_{\tilde{V}}(\tilde{V}) = \varepsilon .$$

Our thickened integral is always within ε !

For the second step of the proof, we apply mixing of the geodesic flow:

$$\lim_{t\to\infty} \underbrace{\frac{1}{m_K(K)} \cdot \frac{1}{m_{\tilde{V}}(\tilde{V})}} \int_{\tilde{V}} \int_{K} \tilde{f}(\Gamma k v g_t)\, dk\, d\tilde{v}$$

$V = K\tilde{V}$

$$= \lim_{t\to\infty} \frac{1}{m(V)} \int_{\Gamma\backslash SL_2\mathbb{R}} \tilde{f}(\Gamma k v g_t) \cdot \chi_{\Gamma V}(\Gamma k v)\, \underbrace{dk\, d\tilde{v}}_{\text{Haar!}}$$

(this was the point of thickening!)

Let $kv = g$

$$= \lim_{t\to\infty} \frac{1}{m(V)} \int_{\Gamma\backslash SL_2\mathbb{R}} \tilde{f}(\Gamma g g_t)\, \chi_{\Gamma V}(\Gamma g)\, dm_\Gamma$$

$$\xrightarrow[t\to\infty]{\text{mixing}} \frac{1}{m(V)} \cdot \left(\int_{\Gamma\backslash SL_2\mathbb{R}} \tilde{f}(\Gamma g)\, dm_\Gamma \right) \cdot m_\Gamma(\Gamma V) \cdot \frac{1}{m(\Gamma\backslash SL_2\mathbb{R})}$$

V sufficiently small $m_\Gamma(\Gamma V) = m(V)$

$$= \frac{1}{m(\Gamma\backslash SL_2\mathbb{R})} \int_{\Gamma\backslash SL_2\mathbb{R}} \tilde{f}(\Gamma g)\, dm_\Gamma$$

Thus, for t sufficiently large (so that $\left| \text{thickened integral} - \frac{1}{m(\Gamma\backslash SL_2\mathbb{R})} \int_{\Gamma\backslash SL_2\mathbb{R}} \tilde{f}(\Gamma g)\, dm_\Gamma \right| < \varepsilon$)

$$\left| \frac{1}{m_K(K)} \int_K \tilde{f}(\Gamma k g_t)\, dk - \frac{1}{m(\Gamma\backslash SL_2\mathbb{R})} \int_{\Gamma\backslash SL_2\mathbb{R}} \tilde{f}(\Gamma g)\, dm_\Gamma \right| < 2\varepsilon,$$

as desired. $\blacksquare$

$\longrightarrow$ Counting lattice points coming soon!
(Hopefully by Wednesday next week...)