Lecture #7

In the previous lecture, we introduced Lie groups and their Lie algebras and touched on a few of their basic properties. At the end of the lecture, we focused in on a particular example: $SL_2\mathbb{R}$. We saw that $SL_2\mathbb{R}$ is a 3-dimensional (real) Lie group, we found a basis for the Lie algebra $\mathfrak{sl}_2\mathbb{R}$, we computed one parameter subgroups of $SL_2\mathbb{R}$, and noticed that $A = \{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \mid t \in \mathbb{R} \}$ and both $N = \{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in \mathbb{R} \}$ and $N^- = \{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \mid x \in \mathbb{R} \}$ interact in a curious way. Moreover, the importance of this curious property came to light when we introduced the Mautner phenomenon. We saw, as an application of the Mautner phenomenon, the following corollary:

<u>COR</u> Let $\pi: SL_2\mathbb{R} \longrightarrow \mathcal{U}(\mathcal{H})$ be a strongly continuous representation of $SL_2\mathbb{R}$ on a Hilbert space $\mathcal{H}$. Let $v \in \mathcal{H}$ be such that $\pi(g_t)v = v$ for $\forall g_t \in A$. Then v is invariant under <u>all</u> of $SL_2\mathbb{R}$.

Before moving on, we want to (re-)emphasize the following relationship between A and N (and, also, A and N^-). Namely, N and N^- are both normalized by A. In fact, the following formulas are of independent interest when one studies the actions of these one-parameter subgroups:

$$\text{①}\quad \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} 1 & xe^{2t} \\ 0 & 1 \end{bmatrix}$$

$$\text{②}\quad \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ xe^{-2t} & 1 \end{bmatrix}.$$

Indeed, we will come back to these formulas at the end of the lecture as they are related to renormalization dynamics.

First, though, we will study actions of $SL_2\mathbb{R}$ via the representation theory perspective and from this, reap important dynamical information regarding actions of $SL_2\mathbb{R}$. Our primary goal is to prove the Howe-Moore theorem for $SL_2\mathbb{R}$, and, time permitting, $SL_n\mathbb{R}$, and eventually semi-simple Lie groups (although, for the latter, we may end up pointing towards references).

<u>THM</u> (Howe-Moore Vanishing Theorem)

Let $\pi: SL_2\mathbb{R} \longrightarrow \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation of $SL_2\mathbb{R}$ on a Hilbert space $\mathcal{H}$. Assume that the representation has <u>no</u> invariant vectors (i.e. no $v \in \mathcal{H}$ such that for any $g \in G$, $\pi(g)v = v$). Then all matrix coefficients of π vanish at ∞.

<u>NOTE</u> If $SL_2\mathbb{R} \curvearrowright (X,\mu)$, and $\pi: SL_2\mathbb{R} \longrightarrow \mathcal{U}(\underbrace{L^2_0(X,\mu)}_{(\mathbb{C}\cdot\mathbb{1})^\perp})$, the associated unitary representation, has no invariant vectors, then $SL_2\mathbb{R} \curvearrowright (X,\mu)$ is ergodic. Howe-Moore tells us that if $SL_2\mathbb{R} \curvearrowright (X,\mu)$ is ergodic, then, in fact, it must be mixing! Moreover, if we restrict the action to A, $A \curvearrowright (X,\mu)$ is also mixing. Indeed, one need only check that $g_t \to \infty$ as $t \to \pm\infty$.

To start the proof, we need to observe that $SL_2\mathbb{R}$ admits a decomposition which will allows us to characterize sequences $g_n \to \infty$ in $SL_2\mathbb{R}$.

THM Let $g \in SL_2\mathbb{R}$. Then $g = k_1 a k_2$ for some $k_1, k_2 \in SO(2) =: K$ and $a \in A^+ = \{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \mid t \geq 0 \}$.

We call this the Cartan decomposition of $SL_2\mathbb{R}$, and write $SL_2\mathbb{R} = KA^+K$.

NOTE The decomposition is not unique.

EXERCISE The map $KA^+K \to SL_2\mathbb{R}$ is $n:1$. What is n? (This exercise might be easier at the end of the lecture notes.)

pf sketch (Cartan Decomposition): Let $g \in SL_2\mathbb{R}$ and recall the polar decomposition of a matrix: $g = kp$ for k orthogonal and p positive-definite symmetric matrix. Indeed, $p = \sqrt{g^t g}$, where the square root is the positive square root of $g^t g$. (We can take the positive square root by diagonalizing $g^t g$. Note that $g^t g$ is symmetric, and since g has determinant 1, so does $g^t g$. Once $g^t g$ is diagonalized, we take the positive square root of the eigenvalues in the diagonal matrix, call the new matrix $\sqrt{D}$. Then, conjugate by the eigenvectors to get $\sqrt{g^t g}$.)

Note that $p = \sqrt{g^t g}$ also has determinant 1. Now, construct $k = g p^{-1}$, and note that k has determinant 1. Moreover, one can show that k is orthogonal.

<u>EXERCISE</u> Confirm k is orthogonal. Recall $k^t k = Id$ for $k \in O(2)$.

In fact, $k \in SO(2)$ since $\det(k) = 1$.

Now, let's focus on p. Since p is a positive-definite, symmetric matrix, when we diagonalize p, the diagonal matrix has positive eigenvalues. Moreover, we can pick an orthonormal basis of eigenvectors, so

$$p = \tilde{k}_1 a \tilde{k}_1^{-1}$$

for some $a \in A^+$ and $k_1 \in SO(2) = K$. Then

$$g = k \tilde{k}_1 a \tilde{k}_1^{-1}.$$

Setting $k_1 = k\tilde{k}_1$ and $k_2 = \tilde{k}_1^{-1}$, we get the desired representation of g.

$\blacksquare$

NOTE The same argument works for matrices $g \in SL_n\mathbb{R}$, with appropriate definitions of $K := SO(n)$ and A^+. However, for semisimple Lie groups, we have to use the Lie algebra to see that the decomposition holds. See Bekka-Mayer, Ch. 3. Note that we require the appropriate definition of A^+. Moreover, K will be compact.

As a consequence of the Cartan decomposition, we have the following:

LEMMA 1 Let $\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation of a lcsc group admitting a Cartan decomposition KA^+K. Suppose that for all matrix coefficients $\varphi_{u,v}(g) := \langle \pi(g)u, v\rangle$ $\forall\, u,v \in \mathcal{H}$ and for all sequences $\{a_n\}_{n\in\mathbb{N}} \subseteq A^+$ with $a_n \longrightarrow \infty$, one has

$$\lim_{n\to\infty} \varphi_{u,v}(a_n) = 0.$$

Then all matrix coefficients of π vanish at infinity.

NOTE The Lemma tells us that the "data" of going to infinity is encoded in the a_n sequences. Essentially, any $g_n \to \infty$ can be reduced to some $a_n \to \infty$ ... we will see how in the proof!

pf: Assume that there exists $\{g_n\}_{n\in\mathbb{N}} \subset G$ s.t. $g_n \to \infty$ and $\lim_{n\to\infty} \varphi_{u,v}(g_n) \neq 0$ for some $u,v \in \mathcal{H}$. We will show that there exists some $a_n \to \infty$ such that $\varphi_{\tilde{u},\tilde{v}}(a_n) \neq 0$ for some $\tilde{u}, \tilde{v} \in \mathcal{H}$.

Let $g_n = k_n a_n h_n$ for $k_n, h_n \in K$, $a_n \in A^+$ be a Cartan decomposition of g_n. Since $g_n \to \infty$, and $k_n, h_n \in K$ which is compact, we must have that $a_n \to \infty$.

Moreover, since K is compact, by passing to a subsequence, we can assume $h_{n_k} \to h$, and consequently, by strong continuity, $\pi(h_{n_k})u \to \tilde{u} \in \mathcal{H}$. (in norm).

Similarly, by passing to a further subsequence, we can assume that $k_{n_{k_j}}^{-1} \to k$, and $\pi(k_{n_{k_j}}^{-1})v \to \tilde{v} \in \mathcal{H}$ (in norm).
In other words, $\pi(h_{n_{k_j}})u \to \tilde{u}$ and $\pi(k_{n_{k_j}}^{-1})v \to \tilde{v}$.

Now, notice that

$$\varphi_{u,v}(k_{n_{k_j}} a_{n_{k_j}} h_{n_{k_j}}) = \langle \pi(k_{n_{k_j}}) \pi(a_{n_{k_j}}) \pi(h_{n_{k_j}}) u, v \rangle$$

$$= \langle \pi(a_{n_{k_j}})\pi(h_{n_{k_j}})u,\ \pi(k_{n_{k_j}}^{-1})v \rangle$$

$$- \langle \pi(a_{n_{k_j}})\pi(h_{n_{k_j}})u,\ \tilde{v} \rangle + \langle \pi(a_{n_{k_j}})\pi(h_{n_{k_j}})u, \tilde{v} \rangle$$

$$- \langle \pi(a_{n_{k_j}})\tilde{u}, \tilde{v} \rangle + \langle \pi(a_{n_{k_j}})\tilde{u}, \tilde{v} \rangle$$

$$= \Big(\langle \pi(a_{n_{k_j}})\pi(h_{n_{k_j}})u,\ \pi(k_{n_{k_j}}^{-1})v \rangle - \langle \pi(a_{n_{k_j}})\pi(h_{n_{k_j}})u, \tilde{v} \rangle \Big)$$

$$+ \Big(\langle \pi(a_{n_{k_j}})\pi(h_{n_{k_j}})u, \tilde{v} \rangle - \langle \pi(a_{n_{k_j}})\tilde{u}, \tilde{v} \rangle \Big)$$

$$+ \langle \pi(a_{n_{k_j}})\tilde{u}, \tilde{v} \rangle .$$

As j goes to ∞, the only possible remaining term is $\lim_{j\to\infty} \langle \pi(a_{n_{k_j}})\tilde{u}, \tilde{v} \rangle$,

so if $\lim_{n\to\infty} \varphi_{u,v}(g_n) = \lim_{n\to\infty} \langle \pi(g_n)u, v \rangle \neq 0$, then

$$\lim_{n\to\infty} (\pi(a_n)\hat{u}, \tilde{v}) = \lim_{n\to\infty} \varphi_{\tilde{u},\tilde{v}}(a_n) \neq 0.$$

$\blacksquare$

Now, if we can show that $\varphi_{u,v}(a_n) \xrightarrow{n\to\infty} 0$ for all sequences $a_n \to \infty$, we get for free (or rather, as a result of the Cartan Decomposition) that all matrix coefficients must vanish at ∞.

Let's pause and try to sketch out a proof of the Howe-Moore vanishing theorem for $SL_2\mathbb{R}$.

(The outline will seem reasonable, but as we continue filling in details, we will have to make a few modifications.)

Assume $\pi : SL_2\mathbb{R} \longrightarrow \mathcal{U}(\mathcal{H})$ has no non-zero invariant vectors.

Fix any $u \in \mathcal{H}$, and any $a_n \longrightarrow \infty$ for $\{a_n\} \subset A^+$.

Observe that since $\pi(a_n)$ is unitary, and $\mathcal{H}$ is a Hilbert space, $\{\pi(a_n)u\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is bounded in norm (by $\|u\|$) and consequently contains accumulation points (by Banach-Alaoglu).

Let u_0 be any accumulation point of $\{\pi(a_n)u\}_{n\in\mathbb{N}}$.

Then, u_0 is invariant under $\pi(a_n)$, so (almost) by the Mautner phenomenon, u_0 is invariant under N and N^-, which means u_0 is invariant under $SL_2\mathbb{R}$. By our assumption that π has no non-zero invariant vectors, we have that $u_0 = 0$ in $\mathcal{H}$. This means that any weak limit of $\{\pi(a_n)u\}_{n\in\mathbb{N}}$ is 0, and for any $v \in \mathcal{H}$,

$$\varphi_{u,v}(a_n) = \langle \pi(a_n)u, v\rangle \longrightarrow 0.$$

Thus, by Lemma 1, $\forall\, u,v \in \mathcal{H}$, $g_n \to \infty$, $\varphi_{u,v}(g_n) \longrightarrow 0$, so all matrix coefficients vanish.

There are a few issues with this argument.

1. The Mautner phenomenon, as we proved it, does not handle arbitrary sequences. However, it shouldn't be surprising based on the proof we gave that it could.

2. u_0 won't necessarily be invariant under N^- since the sequence is in A^+. We need a new argument to show this.

Other than this, the argument should work. Let's upgrade our Mautner phenomenon:

DEF For a sequence $\alpha = \{a_n\}_{n\in\mathbb{N}} \subset G$, where G is a locally compact group, let

$$U_\alpha^+ := \{ g \in G \mid e \text{ is an accumulation point of } \{a_n^{-1} g a_n\}_{n\in\mathbb{N}} \}$$

and let N_α^+ be the closed subgroup generated by U_α^+.

EXAMPLE For $G = SL_2\mathbb{R}$ and a sequence $\alpha \subset A^+ = \{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \mid t \geq 0 \}$, we have that $\alpha = \{ a_n = \begin{bmatrix} \alpha_n & 0 \\ 0 & \frac{1}{\alpha_n} \end{bmatrix} \}_{n\in\mathbb{N}}$ converges to ∞ iff

$$\lim_{n\to\infty} \alpha_n = \infty.$$

Then, notice that U_α^+ can be computed. Indeed, this is quite similarly to the property we noticed in the last lecture! Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, and compute:

$$\begin{bmatrix} \frac{1}{\alpha_n} & 0 \\ 0 & \alpha_n \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha_n & 0 \\ 0 & \frac{1}{\alpha_n} \end{bmatrix} = \begin{bmatrix} a & b \cdot \frac{1}{\alpha_n^2} \\ c\alpha_n^2 & d \end{bmatrix}.$$

As $\alpha_n \to \infty$, we need $c = 0$, $a = 1$, $d = 1$. Thus

$$U_\alpha^+ = N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}.$$

Since N is a closed one-parameter subgroup, we have that $N_\alpha^+ = N$.

THM (Mautner Phenomenon)

Let G be a locally compact group and $\pi: G \longrightarrow \mathcal{U}(\mathcal{H})$ a strongly continuous unitary representation of G. Let $\alpha = \{a_n\}_{n\in\mathbb{N}}$ be a sequence in G and let $u, u_0 \in \mathcal{H}$ satisfy

$$\lim_{n\to\infty} \pi(a_n) u = u_0.$$

Then $\pi(g) u_0 = u_0$ for all $g \in N_\alpha^+$.

(i.e. accumulation points of $\pi(a_n)$ are fixed vectors under $\pi(g)$ for any $g \in N_\alpha^+$.)

pf: Fix any $g \in N_\alpha^+$.
Let $\{a_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of α such that

$a_{n_k}^{-1} g a_{n_k} \longrightarrow e$. Then, $\forall\, v \in \mathcal{H}$,

$$\left| \langle \pi(g) u_0, v \rangle - \langle u_0, v \rangle \right| \overset{\lim_{k\to\infty} \pi(a_{n_k})u = u_0}{=} \lim_{k\to\infty} \left| \langle \pi(g)\, \pi(a_{n_k})u, v \rangle - \langle \pi(a_{n_k})u, v \rangle \right|$$

$\pi(a_{n_k})$ is unitary!

$$= \lim_{k\to\infty} \left| \langle \pi(a_{n_k}^{-1})\, \pi(g)\, \pi(a_{n_k})\, u, \pi(a_{n_k}^{-1}) v \rangle - \langle u, \pi(a_{n_k}^{-1}) v \rangle \right|$$

$$= \lim_{k\to\infty} \left| \langle \pi(a_{n_k}^{-1} g a_{n_k}) u - u, \ \pi(a_{n_k}^{-1}) v \rangle \right|$$

Cauchy-Schwarz

$$\leq \lim_{k\to\infty} \| (\pi(a_{n_k}^{-1} g a_{n_k}) - Id) u \| \cdot \| v \|$$

$\pi(a_{n_k}^{-1} g a_{n_k}) \to \pi(e)$, + strong continuity.

$$= 0.$$

Thus, $\pi(g) u_0 = u_0$ in $\mathcal{H}$. ▨

So that makes our application of the Mautner Phenomenon work, fixing problem ①. To fix problem ②, we need the following:

LEMMA 2 Let $\pi: SL_2\mathbb{R} \longrightarrow \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation of $SL_2\mathbb{R}$ on a Hilbert space $\mathcal{H}$. Let u_0 be a vector in $\mathcal{H}$ that is invariant under N. Then u_0 is invariant under $SL_2\mathbb{R}$.

pf: We will show that invariance under N implies invariance under all of A (not just A^+). Then, Mautner's Lemma (our first version of the Mautner phenomenon) tells us that u_0 is invariant under N^-, and since $SL_2\mathbb{R}$ is generated by $N \cup N^-$, this finishes the proof.

It remains to show that u_0 invariant under N implies that u_0 is invariant under A.

Consider the following matrix coefficient (which is continuous):

$$\varphi_{u_0,u_0}(g) = \langle \pi(g) u_0, u_0 \rangle \quad , \forall g \in SL_2\mathbb{R}.$$

(Note, this is the positive-definite function associated to u_0.)

φ_{u_0,u_0} has a curious property: it is bi-N-invariant, i.e. constant on every double coset NgN, $g \in SL_2\mathbb{R}$.

Indeed, $\forall\, n_1, n_2 \in N$

$$\varphi_{u_0,u_0}(n_1 g n_2) = \langle \pi(n_1 g n_2) u_0, u_0 \rangle$$

$$= \langle \pi(n_1) \pi(g) \underbrace{\pi(n_2) u_0}_{= u_0}, u_0 \rangle$$

$$= \langle \pi(g) u_0, \underbrace{\pi(n_1^{-1}) u_0}_{= u_0} \rangle$$

$$= \langle \pi(g) u_0, u_0 \rangle$$

$$= \varphi_{u_0,u_0}(g).$$

Now, fix any sequence $\lambda_n \in \mathbb{R}$ s.t. $\lambda_n \neq 0$ but $\lambda_n \xrightarrow{n\to\infty} 0$.

Define

$$g_n = \begin{bmatrix} 0 & -\lambda_n^{-1} \\ \lambda_n & 0 \end{bmatrix} \in SL_2\mathbb{R}.$$

Now, let $a = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \in A$. Observe that

$$\underbrace{\begin{bmatrix} 1 & \alpha\lambda_n^{-1} \\ 0 & 1 \end{bmatrix}}_{\in N} \underbrace{\begin{bmatrix} 0 & -\lambda_n^{-1} \\ \lambda_n & 0 \end{bmatrix}}_{= g_n} \underbrace{\begin{bmatrix} 1 & \alpha^{-1}\lambda_n^{-1} \\ 0 & 1 \end{bmatrix}}_{\in N} = \begin{bmatrix} \alpha & -\lambda_n^{-1} \\ \lambda_n & 0 \end{bmatrix} \begin{bmatrix} 1 & \alpha^{-1}\lambda_n^{-1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & 0 \\ \lambda_n & \alpha^{-1} \end{bmatrix}.$$

(Notice that g_n conjugates to a lower triangular matrix ...)

Then, by continuity

$$\varphi_{u_0,u_0}(a) = \lim_{n\to\infty} \varphi_{u_0,u_0}\left(\begin{bmatrix} \alpha & \\ \lambda_n & \alpha^{-1} \end{bmatrix}\right) = \lim_{n\to\infty} \varphi_{u_0,u_0}\left(\begin{bmatrix} 1 & \alpha\lambda_n^{-1} \\ 0 & 1 \end{bmatrix} g_n \begin{bmatrix} 1 & \alpha^{-1}\lambda_n^{-1} \\ 0 & 1 \end{bmatrix}\right)$$

φ_{u_0,u_0} is bi-N-inv't!

$$= \lim_{n\to\infty} \varphi_{u_0,u_0}(g_n).$$

Since g_n does not depend on a, φ_{u_0,u_0} must be constant on all of A. Moreover, we know the value:

$$\varphi_{u_0,u_0}(a) = \langle \pi(a)u_0, u_0\rangle = \langle \pi(e)u_0, u_0\rangle = \|u_0\|^2.$$

By Cauchy-Schwarz, since $\langle \pi(a)u_0, u_0\rangle = \|u_0\|^2$, $\pi(a)u_0$ and u_0 must be linearly independent, ie. $\exists\, C(a) \in \mathbb{C}$ s.t. $\pi(a)u_0 = C(a)\cdot u_0$. Moreover, we know the constant:

$$\langle \pi(a)u_0, u_0\rangle = \langle u_0, u_0\rangle$$

$$\langle \pi(a)u_0 - u_0, u_0\rangle = 0,$$

and so $\pi(a)u_0 = u_0$ in $\mathcal{H}$, for all $a \in A$. i.e., u_0 is invariant under A which finishes the proof.

$\boxtimes$

Finally, we can put together the pieces to prove the Howe-Moore theorem successfully:

pf: (Howe-Moore vanishing theorem)

Assume $\pi: SL_2\mathbb{R} \longrightarrow \mathcal{U}(\mathcal{H})$ has no non-zero invariant vectors. Fix any $u \in \mathcal{H}$, and any $a_n \to \infty$ for $\{a_n\} \subset A^+$.
Observe that since $\pi(a_n)$ is unitary, and $\mathcal{H}$ is a Hilbert space, $\{\pi(a_n)u\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is bounded in norm (by $\|u\|$) and consequently contains accumulation points (by Banach-Alaoglu).
Let u_0 be any accumulation point of $\{\pi(a_n)u\}_{n\in\mathbb{N}}$. Then, by the Mautner phenomenon, u_0 is invariant under N. (Also, see the example immediately prior to the statement of the Mautner phenomenon to see why $N_\alpha^+ = N$.) By Lemma 2, u_0 is invariant under $SL_2\mathbb{R}$. By our assumption, the only invariant vector is 0, so $u_0 = 0$ in $\mathcal{H}$.

Thus, any weak limit of $\{\pi(a_n)u\}_{n\in\mathbb{N}}$ is 0, so for any $v \in \mathcal{H}$,

$$\varphi_{u,v}(a_n) = \langle \pi(a_n)u, v\rangle \longrightarrow \langle 0, v\rangle = 0.$$

Thus, by Lemma 1, $\forall\, u,v \in \mathcal{H}$, $g_n \to \infty$, $\varphi_{u,v}(g_n) \to 0$, so all matrix coefficients vanish. $\blacksquare$