Lecture #6

In the last lecture, we spent time understanding the definition of weak-mixing for locally compact groups acting by measure-preserving transformations on a probability space. The key takeaway was that $G \curvearrowright (X,\mu)$ is weak-mixing if and only if the diagonal action of G on $(X \times X, \mu \times \mu)$ is ergodic.

We ended the lecture with, finally, the definition of a Lie group (which are locally compact groups with more structure):

DEF A Lie group is a smooth manifold G (without boundary) that is also a group. Moreover, the multiplication map

$$\begin{aligned} m: G \times G &\longrightarrow G \\ (g,h) &\longmapsto gh \end{aligned}$$

and the inversion map

$$\begin{aligned} i: G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are both smooth.

We will pick up here and develop a bit of relevant Lie group theory before turning to the geodesic flow on hyperbolic surfaces.

There are two other maps on any Lie group that will turn out to be very important for us:

(1) Left translation (left multiplication)

$$\begin{aligned} L_g : G &\longrightarrow G \\ h &\longmapsto L_g(h) = gh \end{aligned}$$

(2) Right translation (right multiplication)

$$\begin{aligned} R_g : G &\longrightarrow G \\ h &\longmapsto R_g(h) = hg . \end{aligned}$$

Both of these maps are smooth. Indeed, define a map

$$\begin{aligned} i_g : G &\longrightarrow G \times G \\ h &\longmapsto i_g(h) = (g, h). \end{aligned}$$

This is an inclusion map, so smooth. Now observe that

$L_g = m \circ i_g$,

$$\begin{aligned} G &\xrightarrow{i_g} G \times G \xrightarrow{m} G \\ h &\longmapsto (g, h) \longmapsto gh = L_g(h). \end{aligned}$$

Since m is smooth by assumption that G is a Lie group, we see that L_g is a composition of two smooth maps, hence smooth.

<u>EXERCISE</u> Change the inclusion map and show that R_g is smooth.

<u>EXAMPLES</u> (1) $GL_n\mathbb{R}$, the set of invertible $n \times n$ matrices with real entries. It is a group under matrix multiplication and it is an open submanifold of $\mathbb{R}^{n^2}$ $(\cong M_{n\times n}(\mathbb{R}))$.
Multiplication is smooth: matrix entries of a product of two matrices are polynomials in the entries of two matrices. Inversion is smooth

by, for example, Cramer's rule.

(2) $GL_n^+(\mathbb{R})$: the set of matrices with in $GL_n\mathbb{R}$ with positive determinant.
Note that $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = 1/\det(A)$, so we see that $GL_n^+(\mathbb{R})$ is a subgroup of $GL_n\mathbb{R}$ (closed under multiplication and inversion). It is also a smooth manifold: it is the preimage of $(0,\infty)$ under the determinant function, hence an open submanifold of $GL_n\mathbb{R}$. Multiplication and inversion are then smooth since they are restrictions of multiplication and inversion on $GL_n\mathbb{R}$.

(3) Similarly, $GL_n\mathbb{C}$.

<u>EXERCISE</u> Justify that $GL_n\mathbb{C}$ is a Lie group.

(4) More generally, $GL(V)$, the set of invertible linear maps from a real or complex vector space V to itself. This is a group under composition. If V has finite dimension n, any basis of V gives us an (smooth) isomorphism of $GL(V)$ with $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$, so $GL(V)$ is a Lie group. The map between any two such isomorphisms is given by $A \longrightarrow BAB^{-1}$ (where B is the transition matrix between the two bases), which is smooth. Thus, the smooth structure on $GL(V)$ does not depend on the choice of basis.

(5) $\mathbb{R}$ and $\mathbb{R}^n$ are abelian Lie groups under addition.

(6) Similarly, $\mathbb{C}$ and $\mathbb{C}^n$.

(7) $\mathbb{R}^*$ is a 1-dimensional (abelian) Lie group under multiplication. (Actually, $\mathbb{R}^* \cong GL(1,\mathbb{R})$).

(8) $S^1 \subseteq \mathbb{C}^*$ is an abelian Lie group under multiplication. (Isomorphic to $\mathbb{R}/\mathbb{Z}$ ...)

(9) $\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{\text{"n-times"}}$ is an n-dimensional abelian Lie group (Iso

(10) Any finite or countable group with the discrete topology is a zero-dimensional Lie group, or a discrete Lie group.

EXERCISE In our definition of a Lie group, we required that two maps be smooth: m and i (multiplication and inversion). Show that these two maps being smooth is equivalent to the map

$$\varphi: G \times G \longrightarrow G$$
$$(g,h) \longmapsto gh^{-1}$$

being smooth.

DEF Let G, H be Lie groups.
A Lie group homomorphism $\varphi: G \longrightarrow H$ is a group homomorphism that is also a smooth map.

Now, recall that the set of smooth vector fields on a manifold M can be given the structure of a Lie algebra. There is a subset of the smooth vector fields on a Lie group that forms an important subalgebra, the Lie algebra corresponding to the Lie group. First, recall

DEF (Lie algebra)
A Lie algebra (over $\mathbb{R}$) is a real vector space $\mathfrak{g}$ endowed with a map called the bracket from $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, usually denoted by $(X,Y) \longmapsto [X,Y]$, that satisfies the following properties for

all $X, Y, Z \in \mathfrak{g}$:

① Bilinearity: For $a, b \in \mathbb{R}$,

$$[aX+bY, Z] = a[X,Y] + b[Y,Z]$$

$$[Z, aX+bY] = a[Z,X] + b[Z,Y].$$

② Antisymmetry: $[X,Y] = -[Y,X]$

③ Jacobi Identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

EXAMPLES

① The space of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket: For $X, Y \in \mathfrak{X}(M)$ ← notation for smooth vector fields on M

$$[X,Y] : C^\infty(M) \longrightarrow C^\infty(M)$$

by

$$[X,Y]f = XYf - YXf.$$

Remember: for $X \in \mathfrak{X}(M)$, we can think of the smooth vector field as a differential operator, where at each point, we take the directional derivative in the direction of the vector associated to the vector field.

② The vector space of $M_{n\times n}$ ($n\times n$ matrices) forms a Lie algebra with the bracket

$$[A,B] = AB - BA.$$

EXERCISE Confirm that ① and ② satisfy the definition of a Lie algebra. (Note that for ①, you need to show that for $X, Y \in \mathfrak{X}(M)$, you need to show that $[X,Y] \in \mathfrak{X}(M)$.)

DEF The lie algebra corresponding to a Lie group G, which we denote $\mathfrak{g}$ (or Lie(G) in some texts), is the vector space of left-invariant smooth vector fields equipped with the Lie bracket defined in EXAMPLE ① above, where left-invariant means that the vector field is invariant under any left translation: $X \in \mathfrak{X}(G)$ is left-invariant if

left-multiplication map

$$\forall g, \tilde{g} \in G, \quad d(L_g)_{\tilde{g}} : T_{\tilde{g}} G \longrightarrow T_{g\tilde{g}} G$$

differential at $\tilde{g}$

satisfies $$d(L_g)_{\tilde{g}}(X_{\tilde{g}}) = X_{g\tilde{g}}$$

vector at $\tilde{g}$ of the smooth vector field X

the vector at $g\tilde{g}$ of the smooth vector field X,

i.e. $X_{g\tilde{g}}$ is precisely the image of $X_{\tilde{g}}$ under $d(L_g)_{\tilde{g}}$.

<u>Facts</u> ① The property of being left-invariant is preserved under the Lie bracket:

$$\forall g \in G, \quad \underbrace{(L_g)_* [X,Y]}_{\substack{\text{pushforward} \\ \text{of } [X,Y] \text{ under} \\ L_g}} \overset{(*)}{=} [(L_g)_* X, (L_g)_* Y] \overset{\text{both } X,Y \text{ are left-invt!}}{=} [X,Y]$$

where $(*)$ is justified by: $\forall g, \tilde{g} \in G$,

$$(L_g)_*\big([X,Y](\tilde{g})\big) = (dL_g)_{\tilde{g}} [X,Y]$$

vector at $\tilde{g}$ of $[X,Y]$.

and

$$\begin{tikzcd} TG \arrow[r, "dL_g"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circ"] & TG \arrow[d, "\pi"] \\ G \arrow[r, "L_g"'] & G \end{tikzcd}$$

$$\begin{aligned}
\big((dL_g)_{\tilde{g}} [X,Y]_{\tilde{g}}\big) &= [X,Y]_{g\tilde{g}} \quad \text{vector at } g\tilde{g} \\
&= (XY - YX)(g\tilde{g}) \quad \text{of the vector field } XY-YX. \\
&= X_{g\tilde{g}} \circ Y_{g\tilde{g}} - Y_{g\tilde{g}} \circ X_{g\tilde{g}} \\
&= (dL_g)_{\tilde{g}} X_{\tilde{g}} \circ (dL_g)_{\tilde{g}} Y_{\tilde{g}} - (dL_g)_{\tilde{g}} Y_{\tilde{g}} \circ (dL_g)_{\tilde{g}} X_{\tilde{g}} \\
&= [(L_g)_* X, (L_g)_* Y].
\end{aligned}$$

(2) Any left-invariant vector field corresponds to a single vector in the tangent space of G at the identity element, T_eG. Indeed, if X is left-invariant, pick $X_e \in T_eG$. For any $v \in T_eG$, set $v = X_e$, then define $X_g := d(L_g)_e(v)$.

<u>EXERCISE</u> Fill in the necessary details to prove the fact.

Fact (2) can be formalized:

<u>Prop</u> Let G be a Lie group. The map $\varepsilon : \mathfrak{g} \to T_eG$ given by $\varepsilon(X) = X_e$ is a vector space isomorphism.

Thus, we can identify the Lie algebra of a Lie group G with the tangent space at the identity of G. Moreover, the Lie bracket restricted to the tangent space at the identity gives us the bracket on T_eG.

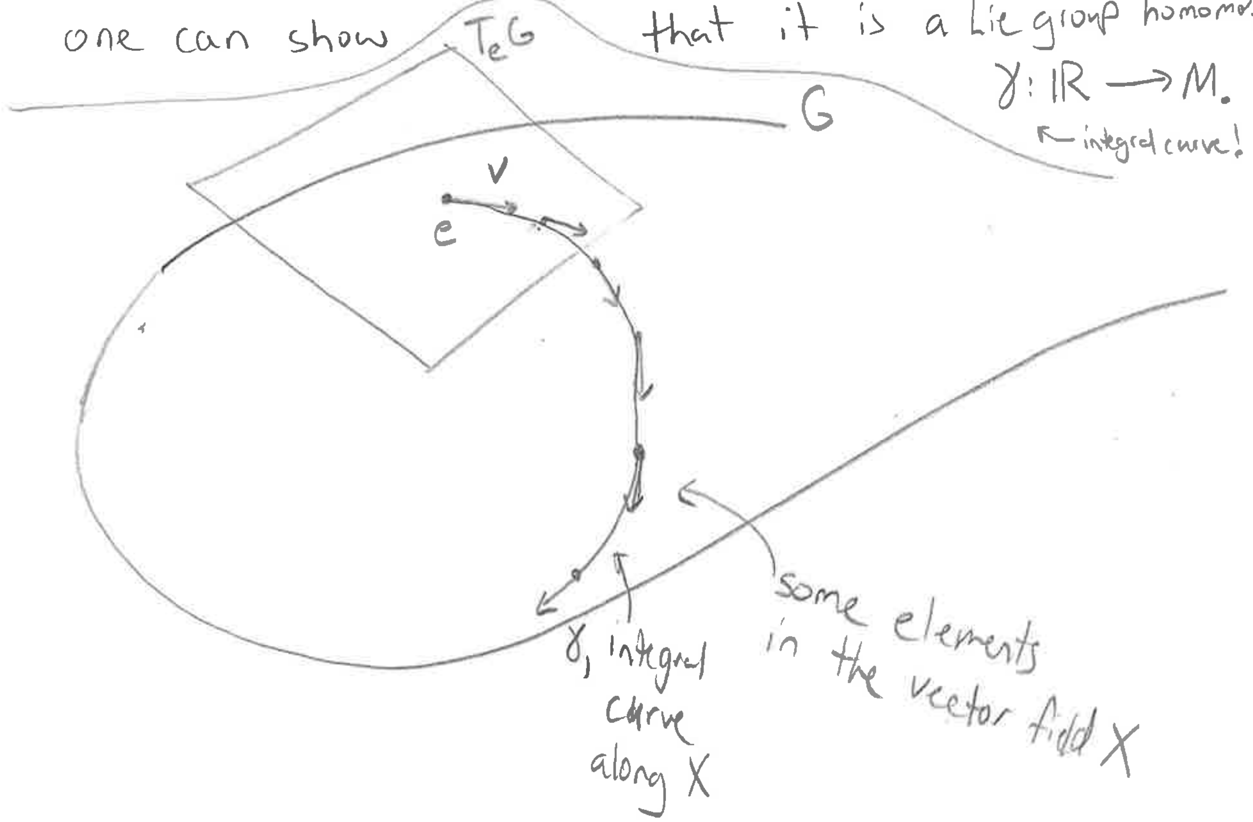
<u>EXAMPLES</u>

(1) Let $G = GL_n\mathbb{R}$. Then $\mathfrak{g} = \mathfrak{gl}_n\mathbb{R} \cong M_{n\times n}$ (the set of $n\times n$ matrices). Indeed, the correspondence is given by

$$T_eG \ni A_{ij} \frac{\partial}{\partial X_{ij}}\bigg|_e \longleftrightarrow A_{ij} \in M_{n\times n}$$

where X_{ij} is the ij^{th} matrix entry, and A_{ij} is the coordinate in $T_eG \cong \mathbb{R}^{n^2}$. To see the details of this correspondence, see J. Lee's text 'Introduction to Smooth Manifolds,' Ch. 8, or Helgason's text 'Differential Geometry, Lie groups, and Symmetric spaces,' Ch. 2.

<u>DEF</u> Let G be a Lie group and $v \in T_eG$. (Then v determines a left-invariant vector field X such that $X_e = v$.) A <u>one-parameter subgroup of G</u> is a maximal integral curve of X, determined by v. More concretely, one can show that it is a Lie group homomorphism $\gamma : \mathbb{R} \to M$.
← integral curve!



The one-parameter subgroups are all isomorphic to either $\{e\}$, $\mathbb{R}$, or S^1.

<u>EXAMPLE</u> Let $G = GL_n\mathbb{R}$. For any $X \in \mathfrak{gl}_n\mathbb{R} \cong \mathfrak{g}$, let

$$e^X := \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

This series converges to an invertible matrix $e^X \in GL_n\mathbb{R}$, and the one-parameter subgroup generated by $X \in \mathfrak{gl}_n\mathbb{R}$ is $\gamma(t) = e^{tX}$.

For instance, let $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $e^{tX} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ and the corresponding one-parameter subgroup generated by X is the group $\left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\}$.

So, we see that the matrix exponential maps $\mathfrak{gl}_n\mathbb{R}$ to $GL_n\mathbb{R}$ and takes each line through the origin $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{gl}_n\mathbb{R}$ to a one-parameter subgroup of $GL_n\mathbb{R}$.

This idea generalizes to arbitrary Lie groups:

<u>DEF</u> Let G be a Lie group with Lie algebra $\mathfrak{g}$. Define a map

$$\exp : \mathfrak{g} \longrightarrow G,$$

the exponential map of G, as follows:

$\forall\, X \in \mathfrak{g}$, define

$$\exp(X) = \gamma(1),$$

where γ is the one-parameter subgroup generated by X.
(Recall, $\gamma : \mathbb{R} \longrightarrow M$ is a Lie group homorphism!)

<u>EXAMPLE</u> The exponential map for $GL_n\mathbb{R}$ is the matrix exponential!

In fact, for any Lie subgroup $H \subseteq G$, $\leftarrow$ Lie group we have that $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(G)$ (and $T_eH \subseteq T_eG$) where we are suppressing the notation of the inclusion map. Moreover, the matrix exponential of G restricted to $\mathrm{Lie}(H) \cong T_eH$ <u>is</u> the exponential map of H! <u>And</u>

$$\mathrm{Lie}(H) = \{ X \in \mathrm{Lie}(G) : \exp(tX) \in H \text{ for all } t \in \mathbb{R} \}.$$

Combining this fact with the prior example, we see that the matrix exponential <u>is</u> the exponential map for any matrix Lie subgroup of $GL_n\mathbb{R}$! i.e., for $SL_n\mathbb{R}$, $SO_n(\mathbb{R})$, $O_n(\mathbb{R})$ ... !

A word of caution: Lie groups and their Lie algebras have many, many more properties (eg. the exponential map, when restricted to a sufficiently small neighborhood of the origin in T_eG, is a diffeomorphism onto a neighborhood of the identity in G). So far, we have barely scratched the surface of this theory. For those interested, there a several good texts on the topic - feel free to reach out and I can recommend one based on your interests.

However, for now, we will turn our attention to one specific Lie group, $SL_2\mathbb{R}$, and eventually think a bit more about $SL_n\mathbb{R}$.

EXAMPLE

- $SL_2\mathbb{R}$ is a Lie group: $SL_2\mathbb{R}$ is a subgroup of $GL_2\mathbb{R}$, and since $SL_2\mathbb{R} = \{ g \in GL_n\mathbb{R} : \det(g) = 1\}$, and the determinant map $\det : GL_n\mathbb{R} \to \mathbb{R}$ is smooth, $SL_2\mathbb{R}$ is a smooth submanifold.

- Its Lie algebra, denoted $\mathfrak{sl}_2\mathbb{R}$, is given by

$$\mathfrak{sl}_2\mathbb{R} = \{ A \in M_{2\times 2} : tr(A) = 0 \}.$$

To see why, observe that the exponential map of $SL_2\mathbb{R}$ is the matrix exponential. Moreover,

$$\det(e^A) = e^{tr(A)}.$$

(<u>EXERCISE</u> If you have not worked out the identity above, try it! (This holds for $n \times n$ matrices, not just 2×2 matrices.))

- $SL_2\mathbb{R}$ is 3-dimensional (over the reals) as a smooth manifold. Indeed, $GL_2\mathbb{R}$ is 4-dimensional since it is an open submanifold of $M_{2\times 2} \cong \mathbb{R}^4$, and $SL_2\mathbb{R}$ has codimension 1 in $GL_2\mathbb{R}$ since $SL_2\mathbb{R} = \{ g \in GL_n\mathbb{R} \mid \det(g) = 1 \}$ and det is smooth.

- This means that its Lie algebra $\mathfrak{sl}_2\mathbb{R}$ is 3-dimensional since $\mathfrak{sl}_2\mathbb{R} \cong T_e(SL_2\mathbb{R})$, and the tangent space at any point in a smooth manifold has the same dimension as the manifold itself.

- It is not hard to justify that $\{H, X, Y\}$ is a basis for the vector space $\mathfrak{sl}_2\mathbb{R}$, where

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

- The lie bracket corresponds to the bracket on matrices, and we can compute:

$$[H,X] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = -2X$$

$$[H,Y] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} = -2Y$$

$$[X,Y] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = H$$

and by properties of the bracket, we have

$$[X,H] = 2X$$
$$[Y,H] = 2Y$$
$$[Y,X] = -H.$$

Indeed, with the first 3 computations, and properties of the Lie bracket, we can compute <u>any</u> bracket!

(**EXERCISE** For $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, compute

$$[a_1 H + a_2 X + a_3 Y,\ b_1 H + b_2 X + b_3 Y].$$)

- Moreover, we can compute the one-parameter subgroups associated to H, X, and Y:

$$\exp tH = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

$$\exp tX = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\exp tY = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

- Let $\begin{cases} N = \{\exp tX \mid t\in\mathbb{R}\} = \left\{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \,\middle|\, x\in\mathbb{R}\right\} \\ N^- = \{\exp tY \mid t\in\mathbb{R}\} = \left\{\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \,\middle|\, x\in\mathbb{R}\right\} \\ A = \{\exp tH \mid t\in\mathbb{R}\} = \left\{\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \,\middle|\, t\in\mathbb{R}\right\}. \end{cases}$

  Then N and N^- are normalized by A.

- Moreover, we have the following curious property:

LEMMA For $g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \in A$, and $h\in N$,

$$\lim_{n\to+\infty} g_t^n h g_t^{-n} = e \quad \text{if } t<0.$$

If $g_t\in A$ and $\tilde{h}\in N^-$, then

$$\lim_{n\to+\infty} g_t^n \tilde{h} g_t^{-n} = e \quad \text{if } t>0.$$

pf: ① $$\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}^n \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}^{-n} = \begin{bmatrix} e^{nt} & 0 \\ 0 & e^{-nt} \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{-nt} & 0 \\ 0 & e^{nt} \end{bmatrix}$$

$$= \begin{bmatrix} e^{nt} & 0 \\ 0 & e^{-nt} \end{bmatrix}\begin{bmatrix} e^{-nt} & xe^{nt} \\ 0 & e^{nt} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & xe^{2nt} \\ 0 & 1 \end{bmatrix}$$

and $xe^{2nt} \longrightarrow 0$ as $n\to+\infty$ if $t<0$. $(\forall x\in\mathbb{R})$.

(**EXERCISE** Confirm the second part of the LEMMA.)

∎

The last property may seem a bit curious, but if we contextualize it in the next Lemma, we see the strength of the property. The following Lemma is a version of the Mautner phenomenon:

LEMMA (Mautner)
Let G be a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation of G on some Hilbert space $\mathcal{H}$. Assume that $g, h \in G$ satisfy

$$\lim_{n \to +\infty} g^n h g^{-n} = e.$$

Then any vector $v \in \mathcal{H}$ that is fixed by g is also fixed by h.

pf: Assume $v \in \mathcal{H}$ is fixed by $\pi(g)$. Then, for any $n \in \mathbb{N}$,

$$\begin{aligned} \|\pi(h)v - v\| &= \|\pi(h)\pi(g^{-n})v - \pi(g^{-n})v\| \\ &= \|\pi(g)\pi(h)\pi(g^{-n})v - \pi(g^n)\pi(g^{-n})v\| \\ &= \|\pi(g^n h g^{-n})v - v\|. \end{aligned}$$

$\pi(g)$ is unitary

Let $n \to \infty$, $g^n h g^{-n} \to e$, and since π is strongly continuous, we have that

$$\|\pi(h)v - v\| = \|\pi(e)v - v\| = \|v - v\| = 0.$$

Thus, we must have that $\pi(h)v = v$.

$\blacksquare$

If we apply this Lemma to $SL_2\mathbb{R}$, we have:

COR Let $\pi: SL_2\mathbb{R} \longrightarrow \mathcal{U}(\mathcal{H})$ be a strongly continuous representation of $SL_2\mathbb{R}$ on a Hilbert space $\mathcal{H}$. Let $v \in \mathcal{H}$ be such that $\pi(g_t)v = v$ for $\forall g_t \in A$. Then v is invariant under all of $SL_2\mathbb{R}$.

pf: Since $g_t^n h g_t^{-n} \longrightarrow e$ as $n \to \infty$ $\forall g_t \in A$, such that $t < 0$ and $\forall h \in N$ and $g_t^n \tilde{h} g_t^{-n} \longrightarrow e$ as $n \to \infty$ $\forall g_t \in A$ such that $t > 0$ and $\forall h \in N^-$, by applying Mautners Lemma in conjunction with the assumption that v is fixed under $\pi(g_t)$ $\forall g_t \in A$, we have that:

$$\begin{cases} \pi(h)v = v & \forall h \in N \\ \pi(\tilde{h})v = v & \forall \tilde{h} \in N^-. \end{cases}$$

Now, observe that $N \cup N^-$ generates $SL_2\mathbb{R}$. Thus $\pi(g)v = v$ for $\forall g \in SL_2\mathbb{R}$.

$\blacksquare$