Lecture #5

In the last lecture, we introduced Haar measures and the notion of a measurable action of a locally compact second countable group. We also defined what it means for a measurable group action to be *ergodic* and *mixing*. In our last example (really, exercise) we outlined a way to show that the action of $SL_n\mathbb{Z}$ on $\mathbb{T}^n$ is *not* mixing despite it being ergodic.

We'll start today's lecture by giving an alternative formulation of mixing, and then move to the notion of a weak-mixing action. We will observe that $SL_n\mathbb{Z}$ acting on $\mathbb{T}^n$ is weak-mixing. After, we will define a Lie group and begin highlighting elements of the structure theory of a Lie group which we will use in a later lecture.

Recall that we said that the measure-preserving action of a locally compact group G on prob. space (X,μ) is mixing if all of the matrix coefficients ($g \xmapsto{\varphi_{u,v}} \langle \pi(g)u,v\rangle$, $u,v \in L^2_0(X,\mu)$) "vanish at infinity," i.e. $\varphi_{u,v} \in C_0(G)$ $\forall\, u,v \in L^2_0(X,\mu)$.

We give an equivalent construction:

<u>DEF</u> Let G be a locally compact group. We say $g_n \to \infty$ if for every compact set $K \subseteq G$, there exists an N such that $g_n \notin K$ $\forall n > N$. In other words, the sequence $\{g_n\}$ "leaves every compact set."

Then, we have

<u>DEF</u> (Alternative) Let G be a locally compact group with a measure-preserving action on a probability space (X,μ). We say the action of G is mixing if for any $u,v \in L^2_0(X,\mu)$ and any $g_n \to \infty$,

$$\langle \pi(g_n)u,v\rangle \longrightarrow 0.$$

<u>Exercise</u> Show that the two definitions of mixing are equivalent. (Recall the notion of "vanish at infinity" from the Lecture #4 notes.)

For the sake of completion, we give one more equivalent definition of mixing and we leave it to the reader to construct a bootstrapping argument to prove equivalence with the other definitions:

<u>DEF</u> (Alternative) With the same setting as prior, we say that a measure-preserving action of G on (X,μ) is mixing if for any $E_0, E_1 \overset{meas}{\subseteq} X$ and any sequence $g_n \to \infty$,

$$\mu(E_0 \cap g_n^{-1} \cdot E_1) \xrightarrow{n\to\infty} \mu(E_0)\mu(E_1).$$

<u>NOTE</u> $\chi_{E_0}, \chi_{E_1} \notin L_0^2(X,\mu)$.

We now turn to the notion of weak-mixing for a measure-preserving action:

<u>DEF</u> Let G be a locally compact group with a measure-preserving action on a probability space (X,μ). The action of G on (X,μ) is <u>weak-mixing</u> if the associated unitary representation $\pi_0 : G \to \mathcal{U}(L_0^2(X,\mu))$, given by

$$\pi_0(g) f(x) = f(g^{-1} \cdot x)$$

where $L_0^2(X,\mu) = (\mathbb{C} \cdot \mathbb{1})^\perp$ (as before), contains no finite-dimensional subrepresentations, that is, there is no G-invariant, non-zero finite-dimensional subspace of $L_0^2(X,\mu)$.

Some remarks are in order. First:

Prop. A mixing measure-preserving action of G, locally compact group, on (X,μ), prob. space, is weak-mixing.

pf: Assume $G \curvearrowright (X,\mu)$ is not weak-mixing. Then there exists a finite-dimensional subspace $V \subseteq L^2_0(X,\mu)$ that is G-invariant. Fix an orthonormal basis $\{v_i\}_{i=1}^n$ of V.

Since $\pi_0(g)$ is unitary $\forall g \in G$, $\|\pi_0(g) v_j\| = 1$ for $\forall g \in G$ and $\forall j \in \{1,\dots,n\}$. Fix a j, and we have that $\pi(g) v_j \in S^{n-1} \subseteq V$ for all $g \in G$.

Let $g_n \longrightarrow \infty$. Then $\{\pi_0(g_n) v_j\}$ is a sequence in S^{n-1} which is compact. Thus, there exists at least one limit point. Call it $w \in S^{n-1}$. Consequently, we can find a subsequence $\{g_{n_k}\}$ such that $\pi(g_{n_k}) v_j \longrightarrow w$ in $V \subseteq L^2_0(X,\mu)$.

Note that since $g_n \to \infty$, we also have that $g_{n_k} \to \infty$.

Consider

$$\langle \pi_0(g_{n_k}) v_j, w \rangle \xrightarrow{k \to \infty} \langle w, w \rangle \neq 0.$$

There exists a matrix coefficient $\varphi_{v_j,w}$ which does not vanish at infinity. Thus $G \curvearrowright (X,\mu)$ is not mixing. ▨

For the reader who has encountered the Peter-Weyl theorem, you may recall that any (strongly) continuous unitary representation of a compact group G decomposes into finite-dimensional irreducible unitary representations.

EXERCISE Use the fact above as a black box to show that no measure-preserving action of a compact group G can be weakly-mixing (and consequently, cannot be mixing either).

Make observation.

Next, we have an equivalent definition of weak-mixing which we will prove here.

THM Let G be a locally compact group and (X, μ) a prob. space. Assume G acts by measure-preserving transformations on (X, μ). Then G is weak-mixing if and only if the diagonal action of G on $(X \times X, \mu \times \mu)$ is ergodic, where the diagonal action is given by

$$g \cdot (x_1, x_2) := (g \cdot x_1, g \cdot x_2).$$

pf: First, assume $G \curvearrowright (X,\mu)$ is weakly-mixing and let $k(x,y) \in L^2(X \times X, \mu \times \mu)$ be G-invariant (with respect to the diagonal action).

WLOG, assume $k(x,y) = \overline{k(y,x)}$, else consider separately $k(x,y) + \overline{k(y,x)}$ and $i(k(x,y) - \overline{k(y,x)})$. (We will show that k is constant a.e., and if both of these expressions are constant a.e., that would imply k is constant a.e.)

Define the following integral operator

$$K : L^2(X,\mu) \longrightarrow L^2(X,\mu)$$

by

$$Kf(x) := \int_X k(x,y) f(y)\, d\mu(y).$$

K is self-adjoint:

$$\langle Kf_1, f_2 \rangle = \int_X \left(\int_X k(x,y) f_1(y)\, d\mu(y) \right) \cdot \overline{f_2(x)}\, d\mu(x)$$

$$\xrightarrow{\text{Fubini}}$$

$$= \int_X \int_X k(x,y) f_1(y) \overline{f_2(x)}\, d(\mu \times \mu)(x,y)$$

$$\downarrow \text{Assumption}$$

$$= \int_X \int_X \overline{k(y,x)}\, \overline{f_2(x)}\, f_1(y)\, d(\mu \times \mu)(x,y)$$

$$= \int_X \left(\int_X \overline{k(y,x) f_2(x)}\, d\mu(x) \right) f_1(y)\, d\mu(y)$$

$$= \int_X \overline{Kf_2}(y) \cdot f_1(y) \, d\mu(y)$$

$$= \langle f_1, Kf_2 \rangle .$$

Moreover, K is Hilbert-Schmidt. Indeed, fix any countable orthonormal basis $\{e_\alpha\}_{\alpha \in \mathcal{A}}$ of $L^2(X, \mu)$ and consider

$$\|K\|_{HS}^2 = \sum_{\alpha \in \mathcal{A}} \|Ke_\alpha\|_2^2 = \sum_{\alpha \in \mathcal{A}} \langle Ke_\alpha, Ke_\alpha \rangle$$

$$= \sum_{\alpha \in \mathcal{A}} \int_X Ke_\alpha(x) \overline{Ke_\alpha(x)} \, d\mu(x)$$

$$= \sum_{\alpha \in \mathcal{A}} \int_X \left(\int_X k(x,y) e_\alpha(y) d\mu(y) \right) \left(\int_X \overline{k(x,y) e_\alpha(y)} \, d\mu(y) \right) d$$

$$= \int_X \int_X k(x,y) \overline{k(x,y)} \, d\mu(y) \, d\mu(x)$$

$$= \|k\|^2 .$$

Since $k \in L_0^2(X \times X, \mu \times \mu)$, $\|K\|_{HS}^2$ is finite. (Note that this makes K not just a bounded operator, but a compact operator, and so K has a nice spectral theory: K can be diagonalized by a countable orthonormal set of eigenvectors, each corresponding to a real eigenvalue. Moreover, each eigenspace is finite-dimensional. We will not prove this, but we will use it as a black box. For a proof, visit Rudin's Functional Analysis or a similar introductory functional analysis text.)

L_0^2 is separa

Now, note that K commutes with $\pi(g)$ for every $g \in G$:

$$\begin{aligned}
\left(K \circ \pi(g)\right) f(x) &= \int_X k(x,y)\, \pi(g) f(y)\, d\mu(y) \\
&= \int_X k(x,y)\, f(g^{-1}y)\, d\mu(y) \\
&= \int_X k(x,gz)\, f(z)\, d\mu(z) \\
&= \int_X k(g^{-1}x, z)\, f(z)\, d\mu(z) \\
&= \left(\pi(g) \circ K\right) f(x).
\end{aligned}$$

let $z = g^{-1}y \Leftrightarrow y = gz$

$d\mu(y) = d\mu(gz) = d\mu(z)$ since $G \curvearrowright (X,\mu)$ is measure-preserv

$\pi(g^{-1}) k(g^{-1}x, y) = k(x, gy)$

k is G-invariant (with respect to the diagonal action)

Consequently, K and $\pi(g)$ are simultaneously diagonalizable and share eigenspaces. Let λ be an eigenvalue and E_λ the corresponding finite-dimensional eigenspace. Since E_λ is an eigenspace for every $\pi(g)$, and it is finite-dimensional, $E_\lambda = \mathbb{C}\cdot\mathbb{1}$. Since this must hold for every λ, K must be a multiple of the orthogonal projection onto $\mathbb{C}\cdot\mathbb{1}$. Thus, $k(x,y)$ must be constant a.e. (equal to this multiple): $\underset{\displaystyle \int_X k(x,y) f(x)\, d\mu(x)}{\underset{\shortparallel}{Kf(x)}} = C \int_X f(x)\, d\mu(x) = \int_X C \cdot f(x)\, d\mu(x)$

So, we can conclude that the diagonal action is ergodic.

For the converse, assume that the diagonal action is ergodic and that there exists a finite-dimensional G-invariant subspace $V \subseteq L^2_0(X,\mu)$. We will show that $V = \{0\}$, showing that $G \curvearrowright (X,\mu)$ is weak-mixing.

Let $\{v_i\}_{i=1}^n$ be an orthonormal basis of V. Define a function $h: X \times X \longrightarrow \mathbb{C}$ by $h(x,y) = \sum_{i=1}^n v_i(x)\overline{v_i(y)}$. Note that $h \in L^2_0(X \times X, \mu \times \mu)$:

$$\iint_{X\,X} h(x,y)\,d\mu(x)\,d\mu(y) = \iint_{X\,X} \sum_{i=1}^n v_i(x)\overline{v_i(y)}\,d\mu(x)\,d\mu(y)$$

$$= \sum_{i=1}^n \Big(\underbrace{\int_X v_i(x)\,d\mu(x)}_{v_i \in L^2_0,\text{ so } \int v_i = 0}\Big)\cdot\Big(\int \overline{v_i(y)}\,d\mu(x)\Big)$$

$$= 0.$$

Let $\pi_0^\Delta : G \longrightarrow \mathcal{U}(L^2_0(X \times X, \mu \times \mu))$ be the unitary representation corresponding to the diagonal action of G. Then,

$$\pi_0^\Delta(g)\,h(x,y) = h(g^{-1}x, g^{-1}y) = \sum_{i=1}^n v_i(g^{-1}x)\overline{v_i(g^{-1}y)}$$

$$= \sum_{i=1}^n (\pi_0(g)v_i)(x)\cdot\overline{\pi_0(g)v_i(y)}$$

$\pi_0(g)$ looks like an $n\times n$ matrix on V

$$= \sum_{i=1}^n \left(\sum_{j=1}^n \langle \pi_0(g)v_i, v_j\rangle v_j(x)\right)$$

$$\cdot \overline{\left(\sum_{k=1}^n \langle \pi_0(g)v_i, v_k\rangle v_k(y)\right)}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \pi_0(g) v_i, v_j \rangle \langle v_k, \pi_0(g) v_i \;\; v_j(x) \overline{v_k(y)},$$

since

$$\overline{\langle \pi_0(g) v_i, v_k \rangle} = \int_X \overline{\pi_0(g) v_i(x) \, \overline{v_k(x)}} \, d\mu(x) = \int_X v_k(x) \cdot \overline{\pi_0(g) v_i(x)} \, d\mu(x).$$

Notice, for any $g \in G$, $\pi(g)$ is unitary and V is G-invariant, so $\{\pi_0(g) v_i\}_{i=1}^{n}$ is also an orthonormal basis of V.

So,

$$\pi_0^{\Delta}(g) h(x,y) = \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \langle \pi_0(g) v_i, v_j \rangle \langle v_k, \pi_0(g) v_i \rangle \right) v_j(x) \overline{v_k(y)}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \big\langle \langle v_k, \pi_0(g) v_i \rangle \pi_0(g) v_i, v_j \big\rangle \right) v_j(x) \overline{v_k(y)}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \langle v_k, v_j \rangle \, v_j(x) \overline{v_k(x)}$$

0 unless $j=k$ and 1 if $j=k$.

$$= \sum_{j=1}^{n} v_j(x) \overline{v_j(y)}$$

$$= h(x,y),$$

and we see that h is G-invariant. Since $h \in L_0^2(X \times X, \mu \times \mu)$, $h = 0$ a.e. and consequently $V = \{0\}$. Thus, $G \curvearrowright (X, \mu)$ is weak-mixing, as desired. ∎

With this Theorem, we can observe that $SL_n\mathbb{Z} \curvearrowright \mathbb{T}^n$ is weak-mixing. In fact, we have almost already done this ...

<u>EXERCISE</u> Prove that $SL_n\mathbb{Z} \curvearrowright \mathbb{T}^n$ is weak-mixing by showing that $SL_n\mathbb{Z} \curvearrowright \mathbb{T}^n \times \mathbb{T}^n$ given by $g \cdot (x,y) := (gx, gy)$ is ergodic. <u>Hint</u>: How did we show that $SL_n\mathbb{Z} \curvearrowright \mathbb{T}^n$ is ergodic? Try a Fourier analysis argument!

This essentially concludes most of the ergodic theory of locally compact group actions that we will need. Any other bits will show up in-line with the rest of the course (including the fact that $SL_2\mathbb{Z} \curvearrowright \mathbb{R}P^1$ has no invariant measures ...)

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§ Lie Groups

<u>DEF</u> A <u>Lie group</u> is a smooth manifold G (without boundary) that is also a group. Moreover, the multiplication map

$$\begin{aligned} m: G \times G &\longrightarrow G \\ (g,h) &\longmapsto gh \end{aligned}$$

and the inversion map

$$\begin{aligned} i: G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are both smooth.