Lecture #1

This course is intended to serve as an introduction to dynamics on Lie groups where throughout the course, our main focus will be on the group $SL_2\mathbb{R}$. We will start with a brief introduction to measurable dynamics (ergodic theory).

By the end of the second lecture, I hope we will have developed all of the necessary (ergodic theoretic) definitions and theorems, not just in the context of iterates of a single map, but for group actions on a probability space. We will move from here to the definition of a Lie group and its corresponding Lie algebra. At this point, we should be prepared to study

1. action of the diagonal subgroup of $SL_2\mathbb{R}$ on $SL_2\mathbb{R}/\Gamma$, where Γ is a lattice in $SL_2\mathbb{R}$. (The definition of a lattice will come, but you may think that $\Gamma \subset SL_2\mathbb{R}$ is something like $\mathbb{Z}^2 \subset \mathbb{R}^2$. Note that $\mathbb{R}^2/\mathbb{Z}^2$ has "finite volume" for a natural choice of volume.)

(2) action of a unipotent subgroup of $SL_2\mathbb{R}$ on $SL_2\mathbb{R}/\Gamma$ (think "horocycle flow".)

(3) action of $SL_2\mathbb{R}$ on $SL_2\mathbb{R}/\Gamma$

and, eventually,

(4) action of a lattice $\Gamma \subset SL_2\mathbb{R}$ on an arbitrary probability space.

There will be a few key technical high points in the course, including (but perhaps not limited too!)

(1) Hopf Argument

(2) Mautner phenomenon

(3) Howe-Moore Theorem (vanishing of matrix coefficients).

The arguments we give may hint at a well-developed theory of harmonic analysis, at least in the setting of $SL_2\mathbb{R}$, but that will not be the focus of this course.

I plan to give at least one key application of the Howe-Moore Theorem, which will involve two things:

① "Equidistribution of circles" in $\mathbb{H}/\Gamma$

② the Hyperbolic lattice point counting problem (non-effective, at least at first).

Time permitting, we may talk a bit about how to make some of these results "effective" or "quantitative," by which I mean understanding error terms associated with ergodic theorems, theorems that show convergence in some way, and asymptotics.

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§ Recurrence & Ergodicity

We start with a definition, a few examples, and a key property that motivates the study of dynamics.

Could say: "finite measure space"

<u>DEF</u> A <u>measure</u> <u>preserving</u> <u>transformation</u> on a probability space $(X, \mathcal{B}, \mu)$ is a measurable map $T: X \longrightarrow X$ such that for any $E \in \mathcal{B}$, $\mu(T^{-1}(E)) = \mu(E)$.

<u>REMARK</u> ① Equivalently, $T_* \mu = \mu$ (the pushforward of the measure μ by T is μ). This means, equivalently, for any $f \in L^2(X, \mu)$, $\int_X f(T(x))\, d\mu = \int_X f\, d(T_*\mu) = \int_X f\, d\mu$.

(see Einsiedler-Ward, Lemma 2.6)

② We don't really need $(X, \mathcal{B}, \mu)$ to be a probability space, but since our dynamics will be happening on a measure space with finite measure, we may as well focus in on probability spaces. (We can get from a finite measure to a measure μ s.t. $\mu(X) = 1$ by renormalizing.)

EXAMPLES

① For any $\alpha \in \mathbb{R}$, define the circle rotation R_α by

$$R_\alpha : S^1 \longrightarrow S^1 \quad \text{"}d\theta \text{ be the angle measure"}$$
$$e^{2\pi i \theta} \longrightarrow e^{2\pi i(\theta + \alpha)}$$

(or, if you prefer..

$$R_\alpha : \overset{\cong \mathbb{T}}{\mathbb{R}/\mathbb{Z}} \longrightarrow \overset{\cong \mathbb{T}}{\mathbb{R}/\mathbb{Z}} \quad \text{"}dm = \text{Lebesgue"}$$
$$t \longmapsto t + \alpha \pmod 1.$$

These maps preserve the natural measures on the space (Lebesgue).

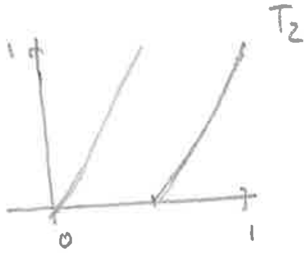
<u>Exercise</u>: Show this! Hint: Consider $\int_{\mathbb{R}/\mathbb{Z}} f \circ R_\alpha \, dm$ Expand f using a Fourier series, and deduce that $\int_{\mathbb{R}/\mathbb{Z}} f \circ R_\alpha \, dm = \int_{\mathbb{R}/\mathbb{Z}} f \, dm$,

i.e. $R_{\alpha *}(m) = m$.

② Observe that $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{Z}^2$ for $\begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{Z}^2$.

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ preserves the integer lattice, so it descends to a map on $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2$. Since $\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right) = 1$, this map preserves the Lebesgue measure on $\mathbb{T}^2$.

③ 2-times map on $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}^1$:

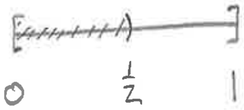


$$T_2 : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$$
$$x \longmapsto 2x \pmod{\mathbb{Z}}.$$

<u>NOTE</u>: Surjective, not injective, but still preserves the Lebesgue measure!

<u>Rem</u> we are interested in pre-images

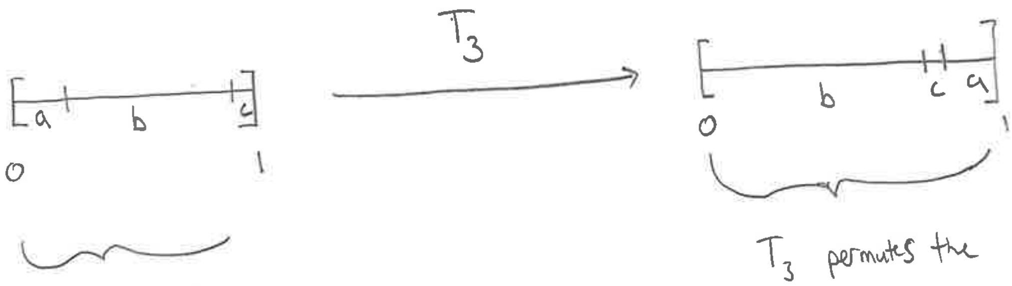
<u>eg</u>



$$T_2^{-1}([0,\tfrac{1}{2})) = [0,\tfrac{1}{4}) \sqcup [\tfrac{1}{2},\tfrac{3}{4}).$$

<u>Exercise</u>: Show this measure is preserved!

④ 3-IET:



divide into "3" pieces a, b, c of lengths ℓ_1, ℓ_2, ℓ_3 where $\ell_1+\ell_2+\ell_3 = 1$

T_3 permutes the pieces a, b, c in a specified way. This map sends (a,b,c) to (b,c,a).

<u>NOTE</u> We can write T_3 as a piecewise function, where each "piece" is just a translation! Lebesgue is invariant under translations.

This preserves the Lebesgue measure on $[0,1]$.

<u>Exercise</u> Using an analogous definition, show that a 2-IET is a circle rotation.

⑤ Bernoulli Shifts (see, for example, Einsiedler-Ward, Examples 2.8, 2.9).

A key question in dynamics revolves around understanding the orbits of points under a transformation. Measurable dynamics seeks to address these questions through measures on the space, whereas topological dynamics seeks to answer these question through topological properties (eg. dense orbits?). Our focus will be primarily developing the measurable perspective (Ergodic theory). We start with a foundational result in measurable dynamics, which is akin to a "pigeon-hole principle" for measure preserving transformations on a probability space.

THM (Poincaré Recurrence)

Let $T: X \to X$ be a measure preserving transformation on a probability space $(X, \mathcal{B}, \mu)$ and let $E \in \mathcal{B}$ be any measurable set. Then a.e. $x \in E$ returns to E infinitely often:

$\exists\ F^{\text{meas}} \subseteq E$ with $\mu(F) = \mu(E)$ such that for every $x \in F$, $\exists$ integers $0 < n_1 < n_2 < \dots$ such that $T^{n_i}(x) \in E$ for all $i \geq 1$.

In other words, the "finite measure" space is "small" enough that almost all points in measurable sets recur (to the starting set) infinitely often under the transformation T.

pf: Let $B = \{x \in E \mid T^n(x) \notin E \ \forall n \geq 1\}$, the set of

$\uparrow$ B for "bad set"

points in E that never return. We can write B

as

$$B = E \cap \underbrace{T^{-1}(X \setminus E)}_{\substack{y \in T^{-1}(X\setminus E) \\ \Leftrightarrow T(y) = X\setminus E, \\ \text{i.e. these are} \\ \text{the points miss } E \\ \text{under the first} \\ \text{iteration of } T}} \cap \underbrace{T^{-2}(X\setminus E)}_{\substack{\text{points that} \\ \text{miss } E \\ \text{under } 2^{nd} \\ \text{iteration.}}} \cap \cdots$$

Thus, since B is the intersection of measurable sets, B is measurable.

Consider the following sets for $n \geq 1$:

$$T^{-n}(B) = \underbrace{T^{-n}(E)}_{\substack{\text{points in } E \\ \text{after } n \\ \text{iterates}}} \cap T^{-n-1}(X\setminus E) \cap \cdots$$

$\rightarrow$ but never return to E again!

i.e., $T^n(x)$ last time we land in E!

Notice ① $\mu(T^{-n}(B)) = \mu(B)$ since T is measure preserving.

② $\mu(B), \mu(T^{-1}(B)), \ldots, \mu(T^{-n}(B)), \ldots$ are all disjoint! Indeed, if $x \in T^{-n}(B) \cap T^{-k}(B)$, $\forall n, k \geq 0$ where (WLOG) $k < n$, $\underbrace{x \in T^{-k}(B)}_{T^k(x) \text{ last time in } E.} \Rightarrow x \notin \underbrace{T^{-n}(B)}_{T^n(x) \notin E.}$

Then, consider that $\bigsqcup_{n=0}^{\infty} T^{-n}(B) \subset X$. We have

$$1 = \mu(X) \overset{\text{monotonicity}}{\geq} \mu\left(\bigsqcup T^{-n}(B)\right) \overset{②}{=} \sum_{n=0}^{\infty} \mu(T^{-n}(B)) \overset{①}{=} \sum_{n=0}^{\infty} \mu(B),$$

and we can conclude $\mu(B)=0$. i.e. the set of points that never return has measure 0. Consequently, the measure of points that return at least once has full measure in E. Call this set F_1, where $\mu(F_1)=\mu(E)$.

Apply the same argument to the measure preserving transformations $T^2, T^3, \ldots$ and get a full measure set for each map. Call these $F_2, F_3, \ldots \subseteq E$, where

$$\mu(F_2) = \mu(F_3) = \cdots = \mu(E).$$

Consider

$$F := \bigcap_{n \geq 1} F_n \subseteq E$$

For any $x \in F$, $\exists$ a sequence of positive integers $k_1, k_2, \ldots$ s.t.

$$x \in F \iff \begin{cases} x \in E \\ T^{k_1}(x) \in E \\ T^{2k_2}(x) \in E \\ \vdots \\ T^{nk_n}(x) \in E \\ \vdots \end{cases}$$

The claim is that the set $\{nk_n\}_{n\in\mathbb{N}}$ contains infinitely many distinct integers: assume otherwise. Then $\exists\, N\in\mathbb{N}$ s.t. $nk_n \le N$ for any $n\in\mathbb{N}$. Pick $n>N$. Since k_n is a positive integer,

$nk_n > Nk_n \geq N$, a contradiction.

So, points in F land back in E infinitely often. Lastly,

$$\mu(F) = \mu\Big(\bigcap_{n\in\mathbb{N}} F_n\Big) = \mu(E)$$

since each F_n has full measure in E. (To see this, consider

$\mu\Big(\bigcup_{n\geq 1}(E\setminus F_n)\Big) \leq \sum_{n\geq 1}\mu(E\setminus F_n) = 0$. Apply DeMorgan's.) ∎

Poincaré recurrence tells us that in this sufficiently constrained setting, orbits of points are interesting. Already, with few assumptions on the map, we know that a.e. point in a measurable set comes back to that set infinitely often. Naturally, we can ask many more questions: For example

1. If a.e. $x\in E$ recurs under T infinitely often... does any $x\in X$ hit the set E infinitely often? What is special about constraining ourselves to E?

2. If $x\in E$ recurs under T infinitely often, can we say precisely how often? What is the

"return time" to E for each point in the full measure set F? Is it uniformly bounded? Above? Below? Are there certain points in the set F which take unbearably long to return to E? (Interpret unbearable however you wish!)

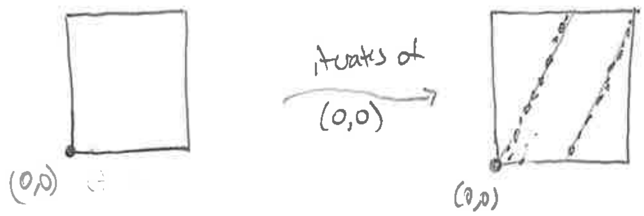
In some sense, it should be obvious that the answers to such questions depend on the map T. However, for a large class of maps, can clarify ①. Consider the following examples which highlight a few ways ① can fail:

- $$R_{1/2} : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$
$$x \longmapsto x + \tfrac{1}{2} \pmod{\mathbb{Z}}.$$

  Each orbit has 2 points! In fact, each point has a period of 2 meaning $T^2(x) = x \ \forall x \in \mathbb{R}/\mathbb{Z}$. So, while every $x \in E \overset{\text{meas}}{\subseteq} \mathbb{R}/\mathbb{Z}$ recurs to E (to itself!) infinitely often, we can have points $y \notin E$ which never hit E!

- $$R_{\pi} \times R_{2\pi} : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \longmapsto \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$
$$(x, y) \longmapsto (x+\pi, y+2\pi) \mod \mathbb{Z}^2$$

  It seems like points in each component move around a lot, but pairs of points seem trapped in a smaller set:



Do you see why we can't find a full measure set of $x \in X$ landing in a specified measurable set infinitely often?

We need a definition which amounts to saying when a measure-preserving transformation is "indecomposable," i.e. points in measurable sets aren't stuck in these sets. (Otherwise, we may as well decompose the dynamical system and study the behavior of orbits in smaller sets!)

DEF A measure-preserving transformation $T: X \longrightarrow X$ of a probability space $(X, \mathcal{B}, \mu)$ is ergodic if for any $B \in \mathcal{B}$,

$$T^{-1}(B) = B \implies \mu(B) = 0 \text{ or } \mu(B) = 1.$$

PROP TFAE:

① T is ergodic.

symmetric difference : 

② $\forall\, B \in \mathcal{B}$, $\mu(T^{-1}(B) \triangle B) = 0 \implies \mu(B) = 0$ or $\mu(B) = 1$

③ $A \in \mathcal{B}$, $\mu(A) > 0 \implies \mu\left(\bigcup_{n=1}^{\infty} T^{-n}(A)\right) = 1$, i.e. you can "move things around enough to "cover" X.

④ $A, B \in \mathcal{B}$, $\mu(A), \mu(B) > 0 \Rightarrow \exists\ n \geq 1$ such that

$$\mu(T^{-n}(A) \cap B) > 0,$$

i.e. for two positive measure sets, iterates of one must eventually "overlap" the other in a nontrivial (measure-theoretic) way.

⑤ For $f: X \to \mathbb{C}$ such that $f \circ T = f$ a.e.,

f must be equal to a constant a.e..

pf: See Einsiedler-Ward, §2.3.

NOTE ① ⟺ ⑤ is a bootstrapping exercise. Ergodicity is a statement about invariant sets, and we can write

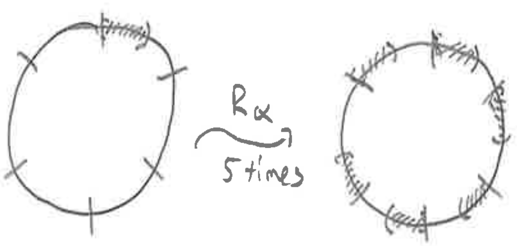
$$T^{-1}(B) = B \Rightarrow \int_X \chi_B \, d\mu = 0 \text{ or } \int_X \chi_B \, d\mu = 1.$$

Prop $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is ergodic w.r.t. the Lebesgue measure m iff α is irrational.

pf: First, let $\alpha \in \mathbb{Q}$, $\alpha = \frac{p}{q}$ in lowest terms. Then $R_\alpha^q = Id$.

Consider the set $A = [0, \frac{1}{2q})$:

eg. $q=6$



then repeats!

Let $B = A \cup R_\alpha(A) \cup R_\alpha^2(A) \cup \cdots \cup R_\alpha^{q-1}(A)$. Then,

B is measurable, B is invariant under R_α, and $m(B) = \frac{1}{2}$.

Since $m(B) \neq 0$ or 1, R_α is not ergodic.

Now, assume $\alpha \notin \mathbb{Q}$.

Let $f \in L^2(\mathbb{R}/\mathbb{Z}, m)$ be a function invariant under R_α. By Prop. 5, it suffices to show f is constant a.e.

Assume $f \circ R_\alpha = f$, let $f(x) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k \cdot x}$ in $L^2(\mathbb{R}/\mathbb{Z}, m)$.

Then,

$$f \circ R_\alpha(x) = f(x+\alpha) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k x} \cdot e^{2\pi i k \alpha}$$

by assumption

$$= \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k x}$$

$$= f(x),$$

Thus, since the fourier transform is an (bijective) isometry on L^2, we need

$$c_k = c_k e^{2\pi i k \alpha} \quad \forall k \in \mathbb{Z},$$

i.e. we need $e^{2\pi i k \alpha} = 1$. But, since $\alpha \notin \mathbb{Q}$, $e^{2\pi i k \alpha} = 1$ iff $k = 0$. So, to have equality, $c_k = 0 \ \forall k \neq 0$, and we see that $f(x) = c_0$ in L^2. Thus $f = c_0$ a.e.

∎

<u>Exercise</u> Using a fourier series, show that when $\alpha \in \mathbb{Q}$, R_α is not ergodic.

As it turns out, ergodicity gives us much more than suggested by the Proposition. In some sense, the definition of ergodic coupled with the Proposition tells us that "an ergodic map moves points around substantially."

But how substantially? The following two theorems give us a bit of insight.

<u>THM</u> (Mean Ergodic Theorem, von Neumann)

Assume $T: X \longrightarrow X$ is ergodic, where $(X, \mathcal{B}, \mu)$ is a probability space. Then, for any $f \in L^2(X, \mu)$,

$$\frac{1}{N} \sum_{n=0}^{n-1} f \circ T^n \xrightarrow[L^2]{N \to \infty} \left(\int_X f \, d\mu \right) \cdot \mathbb{1} .$$

<u>THM</u> (Pointwise Ergodic Theorem, Birkhoff)

Assume $T: X \longrightarrow X$ is ergodic, where $(X, \mathcal{B}, \mu)$ is a probability space. Then, for any $f \in L^1(X, \mu)$,

$$\frac{1}{N} \sum_{n=0}^{n-1} f \circ T^n(x) \xrightarrow{N \to \infty} \int_X f \, d\mu$$

for a.e. $x \in X$.

The first theorem is an L^2-version of the second theorem, and while the second theorem is much harder to prove, it turns out that in most applications, the first theorem is sufficient. Indeed, we will see as the course progresses that it is often sufficient to work in the L^2 setting.

Regardless, the theorems tell us the following: For an ergodic transformation $T : X \longrightarrow X$, the time averages of a function $f \in L^2(X,\mu)$ (or even $L^1(X,\mu)$) converge in some precise sense to the space average of the function:

$$\text{"time average" of } f : \quad \frac{1}{N}\sum_{n=0}^{n-1} f \circ T^n(x)$$

$$\text{"space average" of } f : \quad \int_X f \, d\mu ,$$

i.e., T moves points around enough such that the average $\frac{1}{N}\sum_{n=0}^{n-1} f \circ T^n$ eventually approximates the average of the function over the entire space, $\int_X f \, d\mu$.

Using the pointwise ergodic theorem, we can show how the assumption that T is ergodic allows us to answer ① (following Poincaré Recurrence) positively. Indeed, fix any measurable set $E \in \mathcal{B}$. Then

$$\frac{1}{N} \sum_{n=0}^{n-1} \chi_E \circ T^n(x) \xrightarrow[\text{a.e.}]{} \int_X \chi_E \, d\mu = \mu(E).$$

Notice that $\chi_E \circ T^n(x) = \chi_{T^{-n}(E)}(x)$, so $\frac{1}{N} \sum_{n=0}^{n-1} \chi_{T^{-n}(E)}(x)$ is an average of the number of times that x lands in E (vs. misses). In other words, the pointwise ergodic theorem tells us that a.e. $x \in X$ lands in E infinitely often, and, we have some measure of "how often": x lands in E "$\mu(E)$" of the time. The proportion iterates that land in E amounts to the measure of the set E as we let the number of iterates tend to infinity.

We will not give a proof of the pointwise ergodic theorem, but you can find one in Einsiedler-Ward (§2.6) or Bekka-Mayer (THM 2.5). However, we will give a proof of the Mean Ergodic Theorem in the next lecture, or at least, a version of it.

To prepare for this proof, the remainder of this lecture will be devoted to explaining how we interpret a measure-preserving transformation as an operator on functions. This will enable us to more readily work in function spaces later in the course.

Fix a measure-preserving transformation $T : X \to X$ on a probability space $(X, \mathcal{B}, \mu)$. Define an operator $U_T : L^2(X,\mu) \longrightarrow L^2(X,\mu)$ by

$$U_T(f) = f \circ T \qquad \forall f \in L^2(X,\mu).$$

We call this the <u>Koopman operator</u> or <u>associated operator</u>.

<u>PROP</u> Because T is measure-preserving, U_T is an <u>isometry</u>.

<u>pf:</u> Indeed, for any $f_1, f_2 \in L^2(X,\mu)$,

$$\langle U_T(f_1), U_T(f_2) \rangle = \int_X f_1(T(x)) \cdot \overline{f_2(T(x))} \, d\mu(x)$$

Let $y = T(x)$. Then $d\mu(y) = d\mu(T(x)) = d\mu(x)$

$\nwarrow$ T is measure-preserving!

$$= \int_X f_1(y) \overline{f_2(y)} \, d\mu(y)$$

$$= \langle f_1, f_2 \rangle .$$

Note that for $f = f_1 = f_2$, we have $\|U_T f\|_2 = \|f\|_2$. ▨

Now, assume T is an invertible measure-preserving transformation:

DEF $T: X \longrightarrow X$ is an invertible measure-preserving transformation if T is a measure-preserving transformation, T^{-1} exists a.e., and T^{-1} is a.e. equivalent to a measurable function. (Extend T^{-1} to points where it is not defined however you wish, and this extension must be measurable.)

PROP If T is an invertible measure-preserving transformation, then U_T is a unitary operator.

Recall, an operator U is unitary if $U^* = U^{-1}$, where U^* is the adjoint, and the adjoint is defined by

$$\langle Uf_1, f_2 \rangle = \langle f_1, U^* f_2 \rangle \quad \forall\, f_1, f_2 \in L^2(X, \mu).$$

pf: For T invertible, fix any $f_1, f_2 \in L^2(X, \mu)$. Then

$$\langle U_T f_1, f_2 \rangle = \int_X f_1(T(x))\, \overline{f_2(x)}\, d\mu(x)$$

$\boxed{y = T(x) \iff x = T^{-1}(y)}$

true on a full measure set, so remove the measure 0 set E where this fails, and change variables. Rem: T is measure-preserving, so $d\mu(x) = d\mu(y)$ as before.

$$= \int_{X \setminus E} f_1(y)\, \overline{f_2(T^{-1}(y))}\, d\mu$$

adding the measure 0 set back in doesn't change the integral

$$= \int_X f_1(y)\, \overline{f_2(T^{-1}(y))}\, d\mu$$

$$= \langle f_1, U_{T^{-1}}(f_2) \rangle.$$

Thus, $U_T^* = U_{T^{-1}}$. Now observe that $\forall\, f \in L^2(X, \mu)$,

① $U_{T^{-1}} \circ U_T(f) = f$

and ② $U_T \circ U_{T^{-1}}(f) = f$.

This means that $U_{T^{-1}} = (U_T)^{-1}$, and we have that

$$U_T^* = (U_T)^{-1},$$

so U_T is unitary.

$\boxtimes$

In the case where T is invertible, we call U_T the <u>associated unitary operator</u> (or just the <u>Koopman operator</u>).

It is natural to ask about the spectral properties of the operator U_T: how does the behavior of T, or rather, properties of T, influence the properties of the operator U_T? Here is a sample Lemma that gives a partial answer to this question.

<u>LEMMA</u> Assume $T: X \longrightarrow X$ is a measure-preserving transformation on a probability space $(X, \mathcal{B}, \mu)$. T is ergodic if and only if 1 is a simple eigenvalue of the associated operator U_T.

$\underline{\text{pf:}}$ $U_T(f) = f \overset{\text{def}}{\iff} f \circ T = f$

$\overset{T \text{ is ergodic} + \text{Prop } 5}{\iff}$ f is constant a.e.

Constants are a 1-dim$_{\mathbb{C}}$ subspace in $L^2(X, \mu)$, $\mathbb{C} \cdot \mathbb{1}$. ☒

In the next lecture, we will start with a proof of the mean ergodic theorem, and then explore the notions of mixing and weak-mixing transformations, which are ergodic transformations with stronger dynamical properties. During our discussion of mixing, we'll encounter the Hopf argument. Time permitting, we will begin upgrading all of our definitions to the setting of group actions.