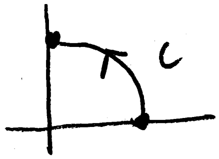
Sample Exam Problem (Sample 3, # 2)

a) Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle x^2 + \frac{1}{y}, e^y \rangle$ and

C is the quarter-(unit circle) traversed counter-clockwise.



STEP 1: Is C closed? (Can you use Green's Theorem)

No

STEP 2: Is $\vec{F}$ conservative? (can you use the fundamental theorem of line integrals?)

No: $\frac{\partial P}{\partial y} = \frac{-1}{y^2} \neq 0 = \frac{\partial Q}{\partial x}$

STEP 3: Solve like a usual line integral.

parametrize C: $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$ $0 \le t \le \frac{\pi}{2}$

$\Rightarrow \vec{r}(t) = \langle \cos t, \sin t \rangle$.

then,

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{\pi/2} \langle \underbrace{(\cos t)}_{x(t)}^2 + \frac{1}{\underbrace{\sin(t)}_{y(t)}}, e^{\overbrace{\sin t}^{y(t)}} \rangle \cdot \overbrace{\langle -\sin t, \cos t \rangle}^{\vec{r}\,'(t)}\, dt$$

$$= \int_0^{\pi/2} \left(-\cos^2 t \sin t - 1 + e^{\sin(t)} \cos t \right) dt$$

$$= \int_0^{\pi/2} \underbrace{(-\cos^2 t \cdot \sin t)\, dt}_{\substack{u = \cos t \\ du = -\sin t\, dt}} - \int_0^{\pi/2} 1 \cdot dt + \int_0^{\pi/2} \underbrace{e^{\sin t} \cos t\, dt}_{\substack{u = \sin t \\ du = \cos t}}$$

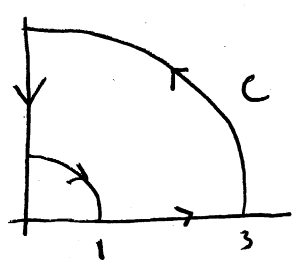
$$= \int_0^{\pi/2} u^2\,du - [t]_0^{\pi/2} + \int_0^{\pi/2} e^u\,du$$

$$= \left[\frac{(\cos t)^3}{3} \right]_0^{\pi/2} - \frac{\pi}{2} + \left[e^{\sin t} \right]_0^{\pi/2}$$

$$= \left(0 - \frac{1}{3}\right) - \frac{\pi}{2} + (e - 1)$$

$$= \boxed{e - \frac{4}{3} - \frac{\pi}{2}}$$

b) Compute $\oint_C (e^x + 6xy)\,dx + (8x^2 + \sin(y^2))\,dy$, where C is the curve consisting of the arcs of the quarter-circle $x^2+y^2=1$ and $x^2+y^2=9$ in the first quadrant:



STEP 1: Is C closed? Yes!

Is C positively oriented? Yes!

Are the partials $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ defined on all of the interior of C (including C) and continuous?

$$\frac{\partial P}{\partial y} = 6x \quad , \quad \frac{\partial Q}{\partial x} = 16x$$

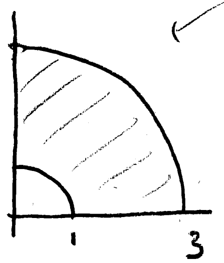
$\Rightarrow$ yes!

We can use Green's Theorem!

Then,

$$\oint_C (e^x + 6xy)\,dx + (8x^2 + \sin(y^2))\,dy = \iint_D (\underbrace{16x}_{\frac{\partial Q}{\partial x}} - \underbrace{6x}_{\frac{\partial P}{\partial y}})\,dA$$

where D is the region "inside" C.



$$= \iint_D 10x\,dA$$

use polar!

$$= \int_{\theta=0}^{\theta=\pi/2} \int_{r=1}^{r=3} 10(r\cos\theta)\,r\,dr\,d\theta$$

$$= 10 \int_0^{\pi/2} \int_1^3 r^2 \cos\theta\,dr\,d\theta$$

$$= 10 \int_0^{\pi/2} \cos\theta \cdot \left[\frac{r^3}{3}\right]_1^3 d\theta$$

$$= 10 \cdot \frac{26}{3} \int_0^{\pi/2} \cos\theta\,d\theta$$

$$= \frac{260}{3} \cdot \left[\sin\theta\right]_0^{\pi/2}$$

$$= \boxed{\frac{260}{3}}$$