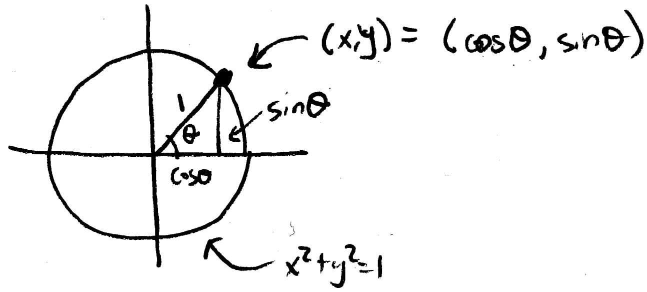
Parametrizing Circles

We start with the parametrization of the unit circle:

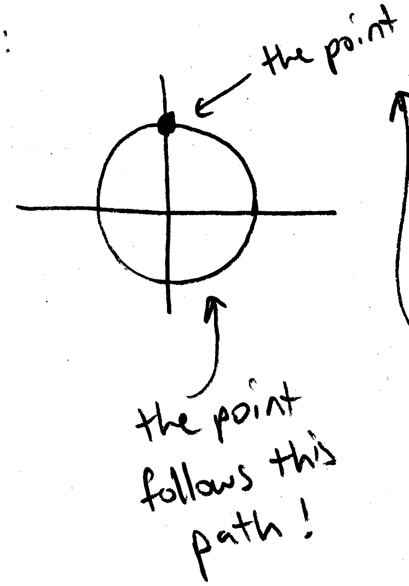
<u>rem</u>:



This is actually a parametrization in terms of θ! We will use t instead of θ since we typically think of a parametrization as giving us
$\begin{cases} \text{1) a point and} \\ \text{2) a path.} \end{cases}$

For example:

the point is at $(0,1)$ at time $t = \pi/2$



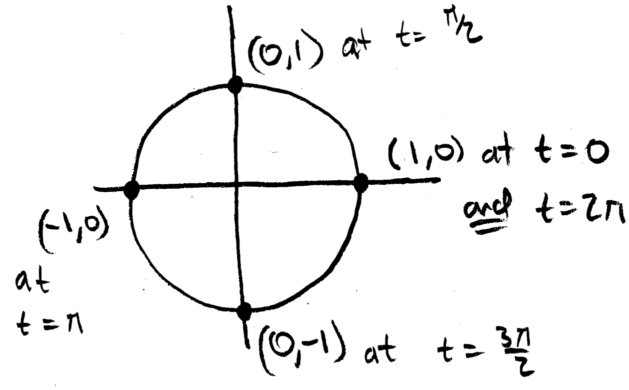
$x(t) = \cos t$

$y(t) = \sin t$

$\begin{cases} \text{at } t = \pi/2, \\ x(\pi/2) = 0 \\ y(\pi/2) = 1 \end{cases}$

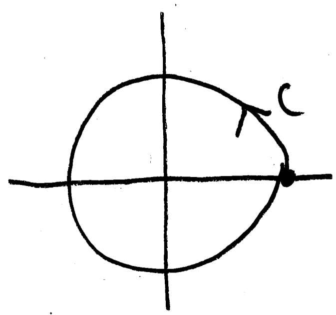
Now, test a few points in time so that you understand how the point moves along the path:

You will find



$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$$

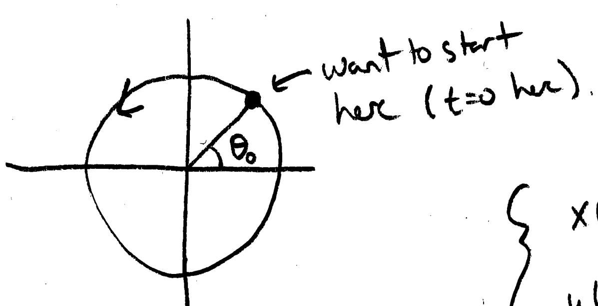
Which suggests we start at (1,0) and follow the curve counter-clockwise (so it is a positively-oriented parametrization)!



$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}, \quad 0 \le t \le 2\pi$$

What if we want to start at a different spot?

Say θ_0:



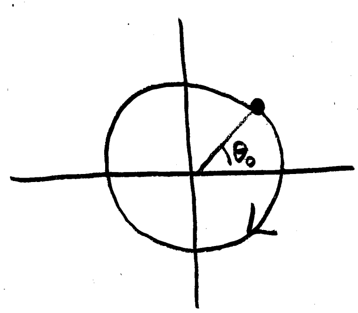
$$\begin{cases} x(t) = \cos(\overbrace{t + \theta_0}^{\text{offset } t \text{ by } \theta_0\,!}) \\ y(t) = \sin(t + \theta_0) \end{cases}$$

We will still follow the curve counterclockwise.

Now, what if we want to change the direction we are traveling, i.e. move clockwise?

There a many ways! (for example, try $x(t) = \sin t$, $y(t) = \cos t$. Where does this start?)

Typically, I think "go backwards in time", so I will put a "$-t$" where we see a "t":



$$x(t) = \cos(-t + \theta_0)$$
$$y(t) = \sin(-t + \theta_0)$$

Try this for $\theta_0 = 0$. Plot $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$.

We can also change the "speed" that our point travels along the path by multiplying t by a number "ω", so $x(t) = \cos(\omega t + \theta_0)$, $y(t) = \sin(\omega t + \theta_0)$

$$\rightsquigarrow \underbrace{\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}}_{\text{takes } 2\pi \text{ "seconds" to get around the circle}} \xrightarrow{\omega = 2\pi} \underbrace{\begin{cases} x(t) = \cos(2\pi t) \\ y(t) = \sin(2\pi t) \end{cases}}_{\text{takes 1 "second" to get around the circle.}}$$

(thinking of t as measured in seconds)

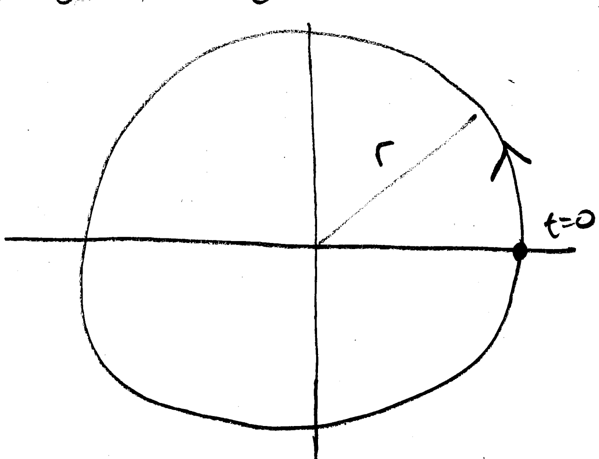
In m124 (if you took it), you learned that if t is in seconds, ω is in radians/sec. So $\omega = 2\pi$ means 2π "radians/sec". Since the circle has 2π radians, it takes 1 second to get around!

For m324, we won't normally need to change the <u>speed</u> of the parametrization, but you should notice that the integral you are evaluating (eg. $\int_C \vec{F} \cdot d\vec{r}$) will be "invariant" under changes to the speed of the parametrization. To see this, compute the integral from Sample Exam 3, #2a, with

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t, \end{cases} \quad 0 \le t \le \pi/2 \quad \text{and} \quad \begin{cases} x(t) = \cos(2\pi t) \\ y(t) = \sin(2\pi t), \end{cases} \quad 0 \le t \le \tfrac{1}{4}.$$

You get the same thing!

Lastly, we may not have a unit circle! What if the radius of the circle is r? (e.g. ... anything but 1)



$$\begin{cases} x(t) = \overbrace{r}^{\text{radius of circle!}} \cos t \\ y(t) = r \sin t \end{cases}$$

(try plotting $t = 0, \pi/2, \pi, \frac{3\pi}{2}, 2\pi$)

If you have an ellipse instead of a circle, all of the same techniques will apply, but we need to notice one thing:

equation of a circle

$$x^2 + y^2 = r^2$$

divide both sides by r^2

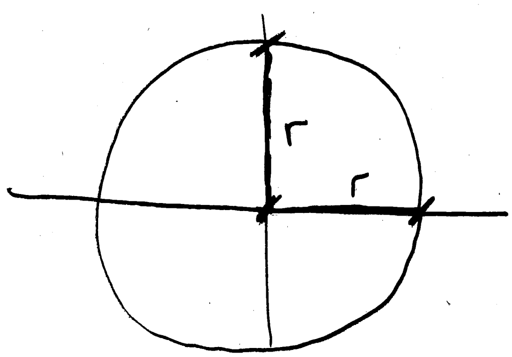
$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

compare!

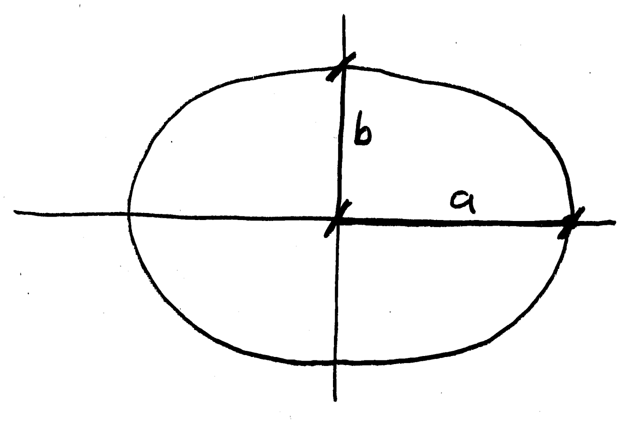
equation of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

picture:



circle



ellipse

$$\begin{cases} x(t) = r\cos t \\ y(t) = r\sin t \end{cases}$$

$0 \le t \le 2\pi$

compare!

$$\begin{cases} x(t) = a\cos t \\ y(t) = b\sin t \end{cases}$$

$0 \le t \le 2\pi$