12.1, 12.2, 12.3, and 12.4 Review

This review sheet discusses, in a very basic way, the key concepts from these sections. This review is not meant to be all inclusive, but hopefully it reminds you of some of the basics. Please notify me if you find any typos in this review.

1. **12.1: Three Dimension Basics**

   (a) *Distance*: The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is
   $$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

   (b) *Circle Equation*: The equation for the sphere (boundary) with center $C(h, k, l)$ and radius r is given by
   $$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

   (c) *Horizontal and Vertical Planes*: The equation $z = 0$ describes the points on the xy-plane in three dimensions. Similarly, $y = 0$ describes the xz-plane and $x = 0$ describes the yz-plane. An equation like $x = 5$ describes a plane parallel to the yz-plane at $x = 5$.

   (d) *Facts for the homework*:

      i. An *isosceles triangle* is a triangle with two sides of the same length.

      ii. The *Pythagorean Theorem* states that a right triangle with legs, a and b, and hypotenuse, c, satisfies $a^2 + b^2 = c^2$. This fact only holds for *right* triangles. So if you have the three lengths of the sides of a triangle, then you can test if it is a right triangle by seeing if $a^2 + b^2 = c^2$.

      iii. To find if *three points are on the same line*, you have a few choices:

         A. You can draw the points and see if you can tell from the graph.

         B. You can read ahead to Section 12.5, where it tells you how to find equations for lines.

         C. You can find the distances between all three points and try to make conclusions from this (In particular, one of the distances will be equal to the sum of the other two if they are on the same line).

2. **12.2: Vector Basics**

   (a) *The Vector Concept*: A *vector* is a quantity that has both magnitude and direction. Typically, we draw a vector as an arrow that starts at the origin. However, as longs as the arrow has the same length and direction, it doesn't matter where we draw it. Here are some standard operations and facts:

      i. *Representations*: We will typically write a vector in one of two ways, *bracket notation*, or *standard basis notation*. They both are essentially the same. For example, $\langle 2, 3, -5\rangle$ is exactly the same as writing $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$. The standard basis vectors are simply $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$.

   (b) *Fundamental Facts*:

   - The two dimensional magnitude of $\mathbf{v} = \langle v_1, v_2\rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$.
   - The three dimensional magnitude of $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.
   - The angle that $\mathbf{v} = \langle v_1, v_2\rangle$ makes with the positive x-axis can be determined by drawing the vector, making a triangle, labeling the angle and using $\tan(\theta) = \frac{\text{opp}}{\text{adj}}$ or some other appropriate trig function.
   - A *unit vector* is a vector with magnitude one. To get a unit vector in the same direction as $\mathbf{v}$, you must divide by the length. That is,
   $$\frac{1}{|\mathbf{v}|}\mathbf{v} = \text{a unit vector in the same direction as } \mathbf{v}.$$
   - *Scalar multiplication*: If c is a scalar and $\mathbf{v}$ is a vector, then $c\mathbf{v}$ means multiply each component of $\mathbf{v}$ by c. This scales the magnitude of $\mathbf{v}$ by a factor c.
   - *Vector Addition*: If $\mathbf{u}$ and $\mathbf{v}$ are two vectors, then $\mathbf{u}+\mathbf{v}$ is the vector obtained by adding the corresponding components of each vector. Graphically, the sum $\mathbf{u} + \mathbf{v}$ is the diagonal of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$. Physically, it can be thought of as the resultant force of the two forces $\mathbf{u}$ and $\mathbf{v}$.

3. **12.3: The Dot Product**: Be able to compute the dot product and use to to find (1) the angle between two vectors and (2) the projection of vectors.

   (a) The *dot product* of the vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is given by
   $$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$
   This definition (and the corresponding results) also work for 2 dimensional vectors.

(b) One particularly nice fact to observe is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

(c) The dot product gives information about the relationship between the two vectors **a** and **b**. In particular, if θ is the angle between these vectors, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta),$$

where $0 \leq \theta < \pi$. Another way to write this is $\cos(\theta) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$, provided neither of the vectors is the zero vector.

(d) Perhaps our biggest application of the dot product is *orthogonality*:

a and **b** are *orthogonal*, or *perpendicular*, if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

(e) *Projections*: To project a vector **b** directly down onto another vector **a**, we use:

$$\begin{aligned} \text{proj}_{\mathbf{a}}(\mathbf{b}) &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} && = \text{vector obtained by projecting } \mathbf{b} \text{ onto } \mathbf{a} \\ \text{comp}_{\mathbf{a}}(\mathbf{b}) &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} && = \text{magnitude of the projection of } \mathbf{b} \text{ onto } \mathbf{a} \end{aligned}$$

4. **12.4: Cross Product**: Understand how to compute the cross product of two (3D) vectors and how to use it to (1) determine if two vectors are parallel and (2) find a vector that is orthogonal to two given vectors.

(a) The *cross product* of two vectors **a** and **b** yields a new vector that is *orthogonal* to both **a** and **b**. In other words, it gives a vector that points out, in a perpendicular way, from the triangle (and plane) determined by **a** and **b**.

(b) The cross product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is given by

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.$$

We can remember how to compute this by either using a determinant or using the diagonal method given in class and quiz section.

(c) The formula looks complicated, but it is defined in this way to ensure that the new vector is orthogonal to both **a** and **b** (That is, you can check $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$).

(d) *Other facts*:

- The area of the parallelogram determined by **a** and **b** is give by $|\mathbf{a} \times \mathbf{b}|$.
- If θ is the angle between **a** and **b** ($0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta).$$

- In particular, the angle is zero (or π) if and only if the vectors are parallel. So the vectors are parallel exactly when $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.