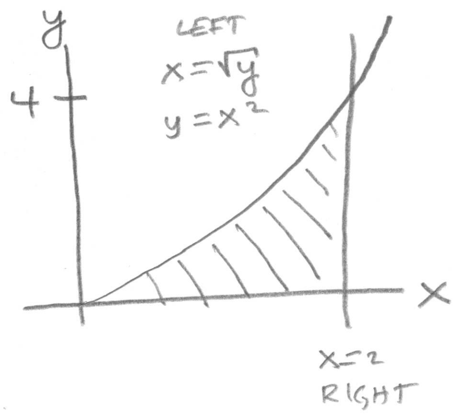
1. (8 pts) Reverse the order of integration and evaluate

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy.$$

LEFT RIGHT

$$\sqrt{y} \le x \le 2$$

$$0 \le y \le 4$$



SWITCH

BOT TOP

$$0 \le y \le x^2$$

$$0 \le x \le 2$$

$$\int_0^2 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx$$

$$\int_0^2 \sqrt{x^3+1}\left[y\Big|_0^{x^2}\right]dx$$

$$\int_0^2 \sqrt{x^3+1}\;x^2\,dx$$

$u = x^3+1$

$du = 3x^2dx$

$dx = \frac{1}{3x^2}du$

$x=0 \longrightarrow u=1$

$x=2 \longrightarrow u=9$

$$\int_1^9 \sqrt{u}\;\cancel{x^2}\frac{1}{3\cancel{x^2}}\,du$$

$$\frac{1}{3}\,\frac{2}{3}\,u^{3/2}\Big|_1^9$$

$$\frac{2}{9}\left(9^{3/2}-1^{3/2}\right) = \frac{2}{9}(27-1)$$

$$= \boxed{\frac{52}{9}}$$

2. (12 pts) Consider the solid region between $z = x$ and $z = x^2$. Let E be the solid that is within this region and bounded between the planes $y = 0$ and $y + 6z = 6$.

(a) Set up the triple integral $\iiint_E 1\, dV$ in each of the specified orders

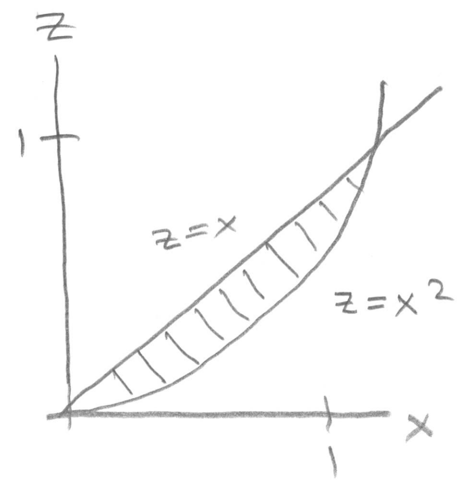
PROJECTION

i. $dydzdx$:

LEFT RIGHT

$\hookrightarrow\ y = 0$ to $y = 6 - 6z$

$\left.\begin{matrix} x^2 \le z \le x \\ 0 \le x \le 1 \end{matrix}\right\} \longrightarrow$



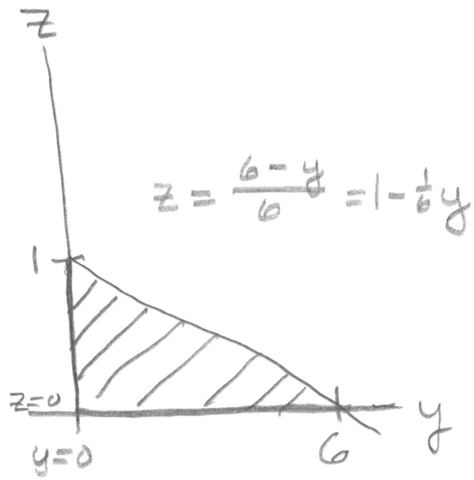
$$\boxed{\int_0^1 \int_{x^2}^{x} \int_0^{6-6z} 1\, dy\, dz\, dx}$$

ii. $dxdzdy$:

BACK FRONT

$\hookrightarrow\ x = z$ to $x = \sqrt{z}$

$\left.\begin{matrix} 0 \le z \le 1 - \frac{1}{6}y \\ 0 \le y \le 6 \end{matrix}\right\} \longrightarrow$



$$\boxed{\int_0^6 \int_0^{1-\frac{1}{6}y} \int_z^{\sqrt{z}} 1\, dx\, dz\, dy}$$

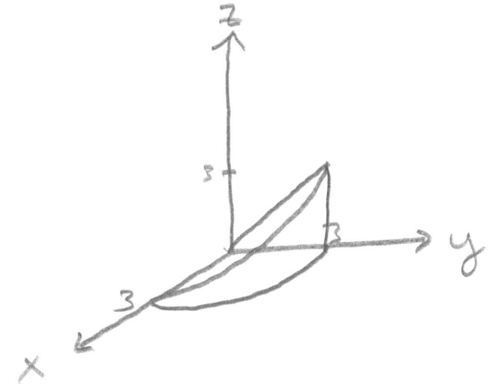
(b) Find the volume of E.

$$\int_0^1 \int_{x^2}^{x} \int_0^{6-6z} 1\, dy\, dz\, dx$$
$$= \int_0^1 \int_{x^2}^{x} 6 - 6z\, dz\, dx$$
$$= \int_0^1 6z - 3z^2 \Big|_{x^2}^{x} dx$$
$$= \int_0^1 (6x - 3x^2) - (6x^2 - 3x^4)\, dx$$
$$= \int_0^1 6x - 9x^2 + 3x^4\, dx$$
$$= 3x^2 - 3x^3 + \tfrac{3}{5}x^5 \Big|_0^1 = 3 - 3 + 3/5$$
$$= \boxed{3/5}$$

CHECK

$$\int_0^6 \int_0^{1-\frac{1}{6}y} \sqrt{z} - z\, dz\, dy$$
$$\int_0^6 \tfrac{2}{3}z^{3/2} - \tfrac{1}{2}z^2 \Big|_0^{1-\frac{1}{6}y} dy$$
$$\int_0^6 \tfrac{2}{3}(1 - \tfrac{1}{6}y)^{3/2} - \tfrac{1}{2}(1 - \tfrac{1}{6}y)^2\, dy$$
$$-6\tfrac{2}{3}\tfrac{2}{5}(1 - \tfrac{1}{6}y)^{5/2} - -6\tfrac{1}{2}\tfrac{1}{3}(1 - \tfrac{1}{6}y)^3 \Big|_0^6$$
$$(0) - (-\tfrac{8}{5} + 1) = \boxed{3/5}\ \checkmark$$

3. (10 points) Let E be the solid bounded in the **first octant** by $x^2 + y^2 = 9$ and $z = y$.
Assume the density of the solid is a constant $\rho(x, y, z) = 6$ kg/m^3.
Use cylindrical coordinates to find the z-coordinate of the center of mass.
(Hint: I'll tell you that the volume of E is 9 m^3).



$$\bar{z} = \frac{\iiint_E 6z\,dV}{\iiint_E 6\,dV} \leftarrow \text{TOTAL MASS}$$

• TOTAL MASS $= 6\underbrace{\iiint_E 1\,dV}_{\text{Volume}} = 6 \cdot 9 = 54 \text{ kg}$

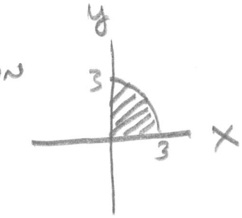
• $\iiint_E 6z\,dV$

Bounds: BOT $0 \le z \le y$ TOP

$0 \le r \le 3$

$0 \le \theta \le \pi/2$

PROJECTION



cylindrical Jacobian

$$\int_0^{\pi/2}\int_0^3\int_0^{r\sin\theta} 6z\; r\,dz\,dr\,d\theta$$

y

$$6\int_0^{\pi/2}\int_0^3 \frac{1}{2}z^2 r\Big|_0^{r\sin\theta} dr\,d\theta$$

$$= 3\int_0^{\pi/2}\int_0^3 r^3\sin^2\theta\,dr\,d\theta$$

$$= 3\left(\int_0^{\pi/2}\sin^2\theta\,d\theta\right)\left(\int_0^3 r^3\,dr\right)$$

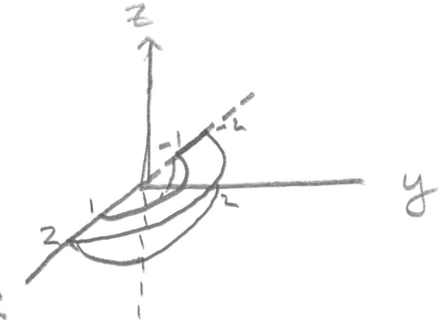
$$= 3\left[\int_0^{\pi/2}\frac{1}{2}(1-\cos(2\theta))\,d\theta\right]\left[\frac{1}{4}r^4\Big|_0^3\right]$$

$$= 3\left[\frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/2}\right]\frac{3^4}{4}$$

$$= 3\left[\left(\frac{\pi}{4} - 0\right) - 0\right]\frac{3^4}{4} = \frac{3^5}{4^2}\pi$$

$$\bar{z} = \frac{1}{54}\cdot\frac{3^5}{4^2}\pi = \frac{1}{\cancel{3^3}\cdot 2}\cdot\frac{\cancel{3^5}^{\,2}}{4^2}\pi = \boxed{\frac{9\pi}{32}}$$

4. (9 points) Let E be the part of the solid bounded between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ with $z \le 0$ and $y \ge 0$.
(In other words, below the xy-plane and on the positive y side of the xz-plane).
Use spherical coordinates to evaluate $\iiint_E \frac{1}{\sqrt{x^2+y^2}}\, dV$

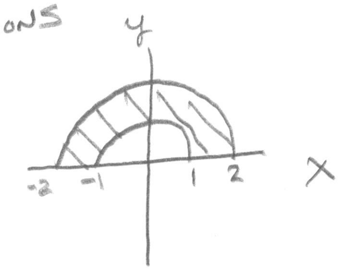


DIST FROM ORIGIN RANGE : $1 \le \rho \le 2$

RANGE OF ANGLES MEASURED DOWNWARD FROM POSITIVE Z-AXIS : $\frac{\pi}{2} \le \phi \le \pi$

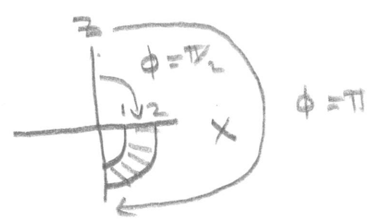
PROJECTIONS

$0 \le \theta \le \pi$



$\sqrt{x^2+y^2} = r$ ← from cylindrical coords

$= \rho \sin\phi$ ← using geometry and as derived in class



$\iiint_E \frac{1}{\sqrt{x^2+y^2}}\, dV$

spherical Jacobian

$= \int_{\pi/2}^{\pi} \int_0^{\pi} \int_1^2 \frac{1}{\rho\sin\phi}\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$

$= \left[\int_{\pi/2}^{\pi} d\phi\right]\left[\int_0^{\pi} d\theta\right]\left[\int_1^2 \rho\, d\rho\right]$

$= \left[\pi - \pi/2\right]\left[\pi - 0\right]\left[\frac{1}{2}(2)^2 - \frac{1}{2}(1)^2\right]$

$= \pi/2 \cdot \pi \cdot \frac{3}{2} = \boxed{\frac{3}{4}\pi^2}$

5. (11 points) Note: Parts (b) and (c) below are unrelated to Part (a).

(a) (3 pts) Compute the Jacobian, $\frac{\partial(x,y)}{\partial(u,v)}$ for the transformation $x = 3u^2 + v^2$ and $y = uv^2$.

$$\begin{vmatrix} 6u & 2v \\ v^2 & 2uv \end{vmatrix} = \boxed{12u^2v - 2v^3}$$

(b) (3 pts) Find the inverse of the transformation: $x = 2u + 2v$ and $y = -2u + 2v$.

$$\begin{array}{rl} x = & 2u + 2v \\ + \quad y = & -2u + 2v \\ \hline x + y = & 4v \end{array}$$

$v = \frac{1}{4}(x+y)$

$$\begin{array}{rl} x = & 2u + 2v \\ - \quad y = & -2u + 2v \\ \hline x - y = & 4u \end{array}$$

$u = \frac{1}{4}(x-y)$

$$\boxed{\begin{array}{l} u = \frac{1}{4}(x-y) \\ v = \frac{1}{4}(x+y) \end{array}}$$

(c) (5 pts) Consider the triangular region, R, in the xy-plane bounded by $(0,0)$, $(4,0)$ and $(4,4)$. A picture of this region is below.
Sketch a detailed graph in the uv-plane of the image of R under the transformation:
$x = 2u + 2v$ and $y = -2u + 2v$.
(Label the new corners and sides).

SIDE 1: $y = 0 \Rightarrow 0 = -2u + 2v$
$\Rightarrow v = u$

SIDE 2: $x = 4 \Rightarrow 4 = 2u + 2v$
$\Rightarrow v = 2 - u$

SIDE 3: $y = x \Rightarrow u = \frac{1}{4}(x - y) = 0$
$\Rightarrow u = 0$