Math 324B
SECOND PRACTICE EXAM SOLUTIONS

1. (a) Abbreviating $\dfrac{\partial f}{\partial u}\left(\dfrac{\tan z}{x^2} + 2\ln y, (2x-3y)^3\right)$ by f_u and likewise for f_v, we have

$$\frac{\partial w}{\partial x} = -\frac{2\tan z}{x^3} f_u + 6(2x-3y)^2 f_v, \quad \frac{\partial w}{\partial y} = \frac{2}{y} f_u - 9(2x-3y)^2 f_v, \quad \frac{\partial w}{\partial z} = \frac{\sec^2 z}{x^2} f_u.$$

(b) Plugging $(x,y,z) = (2,1,\frac{1}{4}\pi)$ and $(f_u, f_v) = (4, -\frac{1}{3})$ into the above formulas gives $\partial w/\partial x = 1$, $\partial w/\partial y = 5$, and $\partial w/\partial z = 2$, so the maximum directional derivative is $|\nabla w| = \sqrt{1^2 + 5^2 + 2^2} = \sqrt{30}$.

2. (a) $\nabla f = (2xy/z - \pi y \sin \pi xy + 2e^{2x-3z})\mathbf{i} + (x^2/z - \pi x \sin \pi xy)\mathbf{j} - (x^2 y/z^2 + 3e^{2x-3z})\mathbf{k}$.
(b) $\nabla f(3,1,2) = (3+0+2)\mathbf{i} + (\frac{9}{2}+0)\mathbf{j} - (\frac{9}{4}+3)\mathbf{k} = 5\mathbf{i} + \frac{9}{2}\mathbf{j} - \frac{21}{4}\mathbf{k}$, so $D_{\mathbf{u}} f(3,1,2) = \frac{9}{2}a - \frac{21}{4}b$. Thus we want $\frac{9}{2}a - \frac{21}{4}b = 0$, or $6a = 7b$. The vectors satisfying this condition are the scalar multiples multiples of $7\mathbf{j}+6\mathbf{k}$, and the ones of unit length are $\mathbf{u} = \pm(7\mathbf{j}+6\mathbf{k})/\sqrt{85}$.

3. The line through $(1,5)$ and $(3,9)$ is $y = 2x+3$. Using x as parameter, we have $ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{5}\,dx$, and the integral is $\int_1^3 \frac{2x+3}{x}\sqrt{5}\,dx = \int_1^3 (2 + \frac{3}{x})\sqrt{5}\,dx = \sqrt{5}[2x + 3\ln x]_1^3 = \sqrt{5}(4 + 3\ln 3)$. Alternatively, you could parametrize the line segment by $x = 1+2t$, $y = 5+4t$, $0 \le t \le 1$. Then $ds = \sqrt{dx^2 + dy^2} = \sqrt{4^2 + 2^2}\,dt = 2\sqrt{5}\,dt$, so the integral is $\int_0^1 [(4t+5)/(2t+1)]\,2\sqrt{5}\,dt$. This reduces to the previous integral by the substitution $x = 2t+1$, or you can do a bit of long division to see that $\frac{4t+5}{2t+1} = 2 + \frac{3}{2t+1}$ and integrate in t directly.

4. Taking $t = y$ as the parameter for the curve, i.e., $\mathbf{r}(t) = (\cos t)\mathbf{i} + t\mathbf{j}$ for $0 \le t \le 2\pi$, we have

$$\int_C (y\mathbf{i} + 2x\mathbf{j}) \cdot d\mathbf{r} = \int_0^{2\pi} [t(-\sin t) + 2\cos t]\,dt = \Big[t\cos t + \sin t\Big]_0^{2\pi} = 2\pi.$$

(By integration by parts, $-\int t \sin t\,dt = t\cos t - \sin t$.)

5. (a) We need (i) $\partial f/\partial x = e^{-2y}$, (ii) $\partial f/\partial y = z^3 - 2xe^{-2y}$, and (iii) $\partial f/\partial z = 3(y+1)z^2$. First, (i) gives $f(x,y,z) = xe^{-2y} + g(y,z)$. Then $\partial f/\partial y = -2xe^{-2y} + \partial g/\partial y$, so (ii) gives $\partial g/\partial y = z^3$, whence $g(y,z) = yz^3 + h(z)$ and so $f(x,y,z) = xe^{-2y} + yz^3 + h(z)$. Therefore $\partial f/\partial z = 3yz^2 + h'(z)$, so (iii) gives $h'(z) = 3z^2$ and hence $h(z) = z^3$. Finally, $f(x,y,z) = xe^{-2y} + (y+1)z^3$ (plus an arbitrary constant).
(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(0,0,2) - f(1,0,1) = (0+8) - (1+1) = 6$.

6. The usual parametrization with $u = x$ and $v = y$, i.e., $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + 2xy\mathbf{k}$, gives $d\mathbf{S} = (\mathbf{r}_x \times \mathbf{r}_y)\,dy\,dx = (-2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k})\,dy\,dx$, which is the upward orientation, so we need to multiply it by -1 for the oriented integral.

(a) We have $dS = |\mathbf{r}_x \times \mathbf{r}_y|\,dy\,dx = \sqrt{4x^2+4y^2+1}\,dy\,dx$ (which doesn't depend on the orientation), so in polar coordinates,

$$A = \int_0^{2\pi} \int_0^2 \sqrt{4r^2+1}\,r\,dr\,d\theta = 2\pi \cdot \tfrac{1}{8} \cdot \tfrac{2}{3}(4r^2+1)^{3/2}\Big|_0^2 = \frac{\pi}{6}(17^{3/2}-1).$$

(b) By the calculation before (a), we have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_{x^2+y^2\le 4} (y\mathbf{i} + x\mathbf{j} + e^{(x+y)^2-2xy}\mathbf{k}) \cdot (2y\mathbf{i} + 2x\mathbf{j} - \mathbf{k})\,dy\,dx \\
&= \iint_{x^2+y^2\le 4} (2y^2 + 2x^2 - e^{x^2+y^2})\,dy\,dx \\
&= \int_0^{2\pi} \int_0^2 (2r^2 - e^{r^2})r\,dr\,d\theta = 2\pi\left[\tfrac{1}{2}r^4 - \tfrac{1}{2}e^{r^2}\right]_0^2 = \pi(17-e^4).
\end{aligned}$$