Solutions. The solutions below for the exam problems refer to pictures on the green handout given out in class on 10/19/11. There are some extra copies outside my office, PDL C338 (in a bin on the wall next to the door).

Problem 1. (a) $\int_0^2 \int_0^{x^2} f(x,y)\,dy\,dx$. (See picture on green handout.)

(b) $\bar{x} = \frac{1}{m}\iint_D xke^y\,dA = \frac{k}{m}\iint_D xe^y\,dA$. The integral is easiest using the set up from part (a):

$$\int_0^2 \int_0^{x^2} xe^y\,dy\,dx = \int_0^2 xe^y\Big|_{y=0}^{y=x^2} dx = \int_0^2 x(e^{x^2}-1)\,dx = \frac{e^{x^2}}{2} - \frac{x^2}{2}\Big|_0^2 = \frac{e^4-5}{2}.$$

(It can also be done in the original order:

$$\int_0^4 \int_{\sqrt{y}}^2 xe^y\,dx\,dy = \int_0^4 \frac{x^2}{2}e^y\Big|_{x=\sqrt{y}}^{y=2} dy = \int_0^4 (2e^y - \frac{ye^y}{2})\,dy = 2e^y - \frac{ye^y}{2} + \frac{e^y}{2}\Big|_0^4 = \frac{e^4-5}{2},$$

using integration by parts to get $\int ye^y dy = ye^y - e^y + C$.) Either way, $\bar{x} = \dfrac{k(e^4-5)}{2m}$.

Problem 2. See picture on handout. If you don't know the polar equation for this circle, find it as follows. The xy-equation for the circle is $(x-2)^2 + y^2 = 4$, or $x^2+y^2 = 4x$, so $r^2 = 4r\cos\theta$, or $r = 4\cos\theta$.

$$\int_{\pi/4}^{\pi/2} \int_0^{4\cos\theta} \frac{k}{r}\, r\,dr\,d\theta = k\int_{\pi/4}^{\pi/2} r\Big|_{r=0}^{r=4\cos\theta} d\theta = 4k\int_{\pi/4}^{\pi/2} \cos\theta\,d\theta = 4k\sin\theta\Big|_{\pi/4}^{\pi/2} = 4k(1-\frac{1}{\sqrt{2}}).$$

(It's not too hard to set up the integral for this mass in xy-coordinates but the integrand is $(x^2+y^2)^{-1/2}$, which is rather complicated to integrate.)

Problem 3. (a) I'm giving a detailed explanation of how to find the limits – more than was required for full credit – because this problem seemed to be the hardest one on the test.

To find the limits y and z, look at region D in the yz-plane which is the projection of the region E to this plane and is bounded by $y = z$, $y^2+z^2 = 1$, and the y-axis. (See picture of D on handout.) In D, the largest value of z occurs where the surfaces $y = z$ and $y^2+z^2 = 1$ intersect, so we get $2z^2 = 1$, or $z = 1/\sqrt{2}$, and the lower limit for z is 0. Still considering D, for fixed z, the lowest value of y occurs on $y = z$ and the highest value on the circle $y^2+z^2 = 1$. Finally, for the limits for x we must think about the three dimensional region E, pictured on the handout. For any point (y,z) in D, the x values are bounded below by the yz-plane, and bounded above by the sphere. Putting all this together, we get

$$\int_0^{1/\sqrt{2}} \int_z^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2-y^2}} f(x,y,z)\,dx\,dy\,dz.$$

Originally, I was going to give you this iterated integral, and ask you to rewrite it as an iterated integral with respect to $dy\,dz\,dx$. Try this, and for even more practice, try several other orders of integration; answers for all but one at the end of this document.

(b) If we integrate first with respect to z, then we have to set limits for z for each choice of (x, y) possible in E. The lower limit for z is always 0, but the upper limit is on the sphere some places and on the plane $y = z$ others. We can see this even just looking at the intersection of E with the yz-plane, which happens to be the same as the projection onto the yz-plane in this example: for $x = 0$ and $y \leq 1/\sqrt{2}$, the upper bound is $z = y$, and for $x = 0$ and larger y it's $z = \sqrt{1 - y^2}$. More precisely, we have an upper limit of $z = y$ in region R_1 in the xy-plane (see picture on handout) and an upper limit $z = \sqrt{1 - x^2 - y^2}$ in region R_2.

Problem 4. The sphere $x^2 + y^2 + z^2 = 2z$ is centered at $z = 1$ on the z-axis and has radius 1. (On the handout there is a picture of the cross section of the two spheres.) In spherical coordinates, it's

$$\rho^2 = x^2 + y^2 + z^2 = 2z = 2\rho\cos\phi,$$

and we can safely cancel one factor of ρ, because the origin remains a solution of the resulting equation $\rho = 2\cos\phi$.

Next we find the ρ-coordinate where the spheres intersect. The other sphere has the equation $\rho = 1$, so they intersect where $\rho = 1 = 2\cos\phi$. This implies $\phi = \pi/3$. Thus the integral is

$$\int_0^{2\pi} \int_0^{\pi/3} \int_1^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

(You can change the order of integration by putting $d\theta$ last, first, or in the middle, but must put $d\rho$ before $d\phi$ unless you want to deal with very messy limits of integration.)

I was going to ask you to compute the volume, but deleted that question because of time. In case you want to try it for practice, here's the final answer: $\frac{11}{12}\pi$.

Problem 5. By the change of variable formula in §15.9,

$$\iint_R \left(\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \right) dA = \iint_S \left(\left(\frac{2u}{2}\right)^2 + \left(\frac{3v}{3}\right)^2 \right) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA,$$

where we need to figure out what region S is in the uv-plane and compute the Jacobian. Substituting, we find S is bounded by

$$1 = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \left(\frac{2u}{2}\right)^2 + \left(\frac{3v}{3}\right)^2 = u^2 + v^2$$

so S is the interior of the unit circle centered at the origin. Pictures of R and S are on the handout. Because they are *different regions*, you should not use the same letter for both of them! The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = 6.$$

Because the transformation is so simple, I did not require that you show the matrix. If you just wrote that $dA = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v}$, be sure you know how to handle cases where none of the entries in the matrix are zero.

To do the integral on S in the uv-plane, it is easiest to use polar coordinates:

$$\iint_S (u^2+v^2)6du\,dv = \int_0^{2\pi}\int_0^1 r^2\,r\,dr\,d\theta = 6\int_0^{2\pi}d\theta\int_0^1 r^3\,dr = 12\pi\frac{r^4}{4}\Big|_0^1 = 3\pi.$$

Other orders of integration for #3.

$$\int_0^{1/\sqrt{2}}\int_0^y\int_0^{\sqrt{1-z^2-y^2}} f(x,y,z)\,dx\,dz\,dy + \int_{1/\sqrt{2}}^1\int_0^{\sqrt{1-y^2}}\int_0^{\sqrt{1-z^2-y^2}} f(x,y,z)\,dx\,dz\,dy$$

$$\int_0^1\int_0^{\sqrt{(1-x^2)/2}}\int_z^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dz\,dx$$

$$\int_0^{1/\sqrt{2}}\int_0^{\sqrt{1-2z^2}}\int_z^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dy\,dx\,dz$$

$$\int_0^1\int_0^{\sqrt{(1-x^2)/2}}\int_z^y f(x,y,z)\,dz\,dy\,dx + \int_0^1\int_{\sqrt{(1-x^2)/2}}^{\sqrt{1-x^2}}\int_z^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

The remaining possibility would have to be written as a sum of three integrals; can you see why?