Quiz #2 Solutions

Set up the following line integrals.

1. $\int_C f(x,y)\,ds$ where C follows the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from $(a,0)$ to $(-a,0)$ in the upper half plane.

parametrize: $\begin{cases} x(t) = a\cos t \\ y(t) = b\sin t \end{cases}$ $0 \le t \le \pi$

(See handout about parametrizing circles!)

(Notice: $\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = 1$)

then,

$$\int_{t=0}^{t=\pi} \overbrace{f(\underbrace{a\cos t}_{x(t)}, \underbrace{b\sin t}_{y(t)})}^{f(x,y)} \cdot \overbrace{\left(\sqrt{(\underbrace{-a\sin t}_{x'(t)})^2 + (\underbrace{b\cos t}_{y'(t)})^2}\right) dt}^{ds}$$

2. $\int_C f(x,y)\,dx$ where C is the line segment from $(1,5)$ to $(7,3)$

parametrize: $\begin{cases} x(t) = at+b \\ y(t) = ct+d \end{cases}$, pick a,b,c,d such that $0 \le t \le 1$

[i.e. $(x(0),y(0)) = (1,5)$
$(x(1),y(1)) = (7,3)$]

$x(0) = b = 1$
$y(0) = d = 5$, then $x(1) = a+1 = 7 \Rightarrow a = 6$
$y(1) = c+5 = 3 \Rightarrow c = -2$

so, we get

$$\begin{cases} x(t) = 6t + 1 \\ y(t) = -2t + 5 \end{cases}$$

then,

$$\int_C f(x,y)\,dx = \int_{t=0}^{t=1} \overbrace{f(\underbrace{6t+1}_{x(t)}, \underbrace{-2t+5}_{y(t)})}^{f(x,y)} \cdot \overbrace{\underbrace{6}_{x'(t)}\,dt}^{dx}$$

3. $\int_C P(x,y)\,dx + Q(x,y)\,dy$ where C is composed of two paths, C_1 and C_2, and

- C_1 follows $x^2+y^2=1$ from $(-1,0)$ to $(1,0)$.
- C_2 follows $y=(x-1)^2$ from $(1,0)$ to $(4,9)$

Parametrize C_1:

$$\begin{cases} x(t) = \cos(-t+\pi) \\ y(t) = \sin(-t+\pi) \end{cases} \quad 0 \le t \le \pi$$

or

$$\begin{cases} x(t) = \cos(-t) \\ y(t) = \sin(-t) \end{cases} \quad -\pi \le t \le 0$$

(see handout/post about parametrizing circles!)

Parametrize C_2:

$$\begin{cases} x(t) = t \\ y(t) = (t-1)^2 \end{cases} \quad 1 \le t \le 4$$

then,

$$\int_C P\,dx + Q\,dy = \underbrace{\int_{C_1} P\,dx + Q\,dy} + \underbrace{\int_{C_2} P\,dx + Q\,dy}$$

$$= \int_0^{\pi} P(\underbrace{\cos(-t+\pi)}_{x(t)}, \underbrace{\sin(-t+\pi)}_{y(t)}) \cdot \underbrace{\sin(t+\pi)}_{x'(t)}\,dt + Q(\overbrace{\cos(-t+\pi)}^{x(t)}, \overbrace{\sin(-t+\pi)}^{y(t)}) \cdot \underbrace{(-\cos(-t+\pi))}_{y'(t)}\,dt$$

$$+ \int_1^4 P(\underbrace{t}_{x(t)}, \underbrace{(t-1)^2}_{y(t)}) \cdot \overbrace{\underbrace{1}_{x'(t)}\,dt}^{dx} + Q(\underbrace{t}_{x(t)}, \underbrace{(t-1)^2}_{y(t)}) \cdot \overbrace{\underbrace{2t}_{y'(t)}\,dt}^{dy}.$$