Quiz

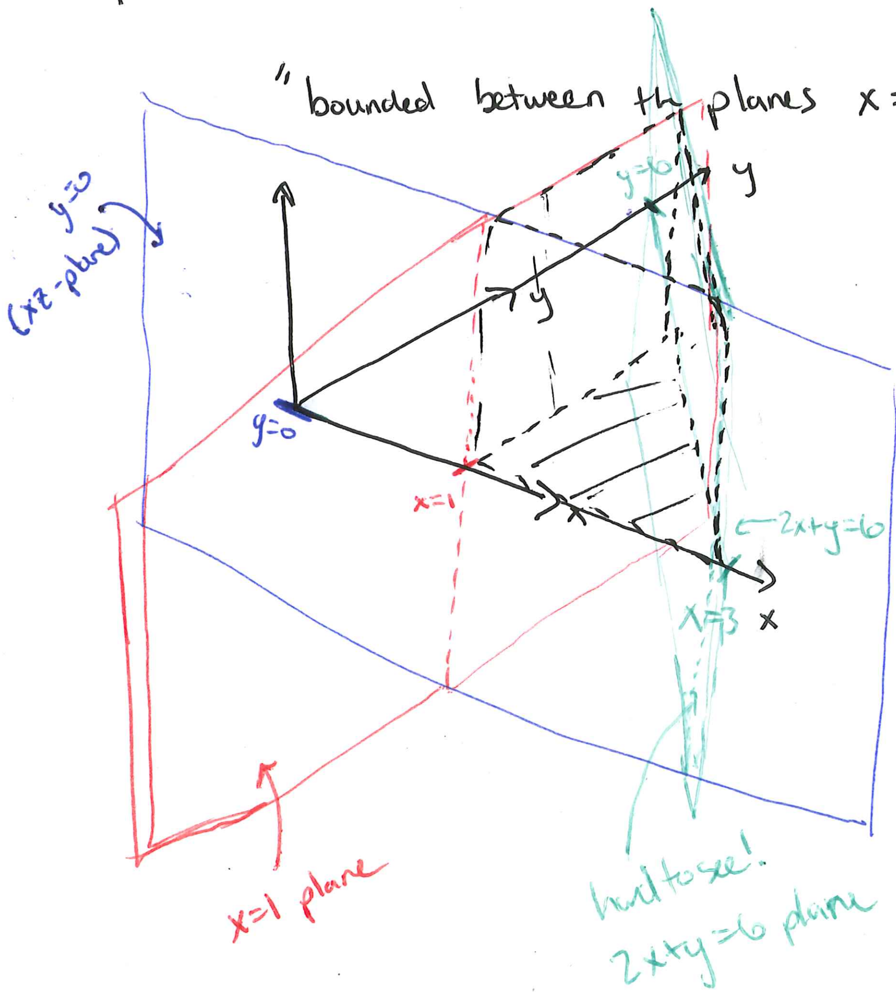
Let E be the solid bounded between the planes $x=1$, $y=0$, and $2x+y=6$, bounded below by $z=0$ and bounded above by $z=\sqrt{x}$.

Set up the integral $\iiint_E f(x,y,z)\,dV$ in three different ways.

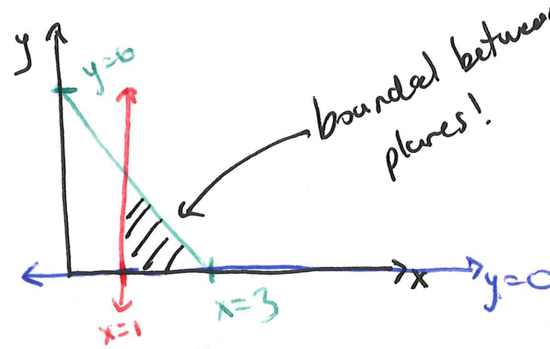
Solution

Step 1: Draw domain.

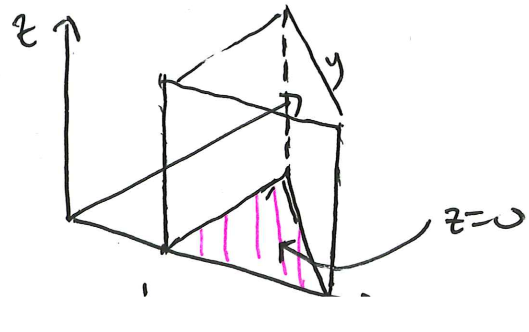
"bounded between the planes $x=1$, $y=0$, and $2x+y=6$"



view from above

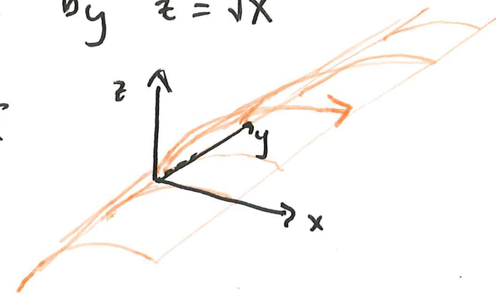


"bounded below by $z=0$"

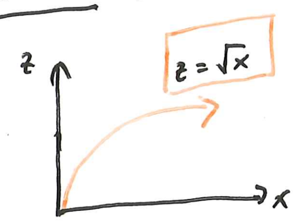


" bounded above by $z = \sqrt{x}$ "

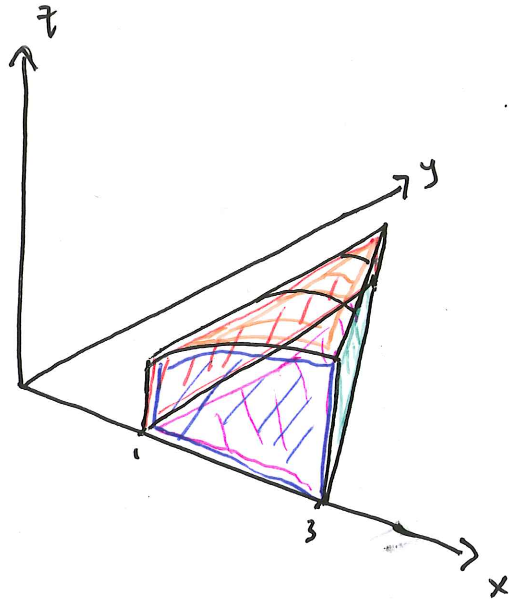
$z = \sqrt{x}$



side view:



so:



$y=0$

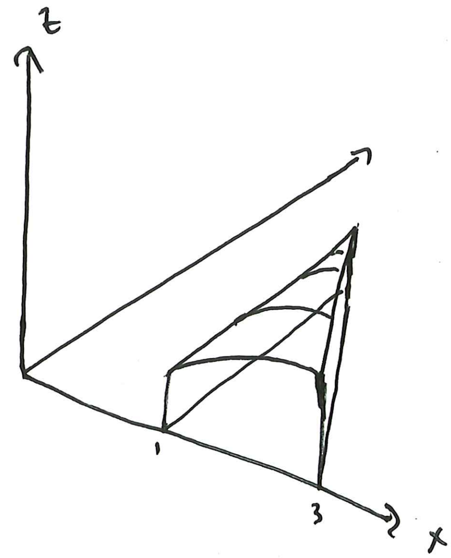
$x=1$

$2x+y=6$

$z=0$

$z=\sqrt{x}$

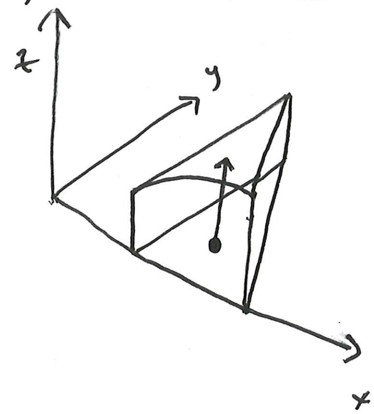
No colors:



Step 2: Setup integrals!

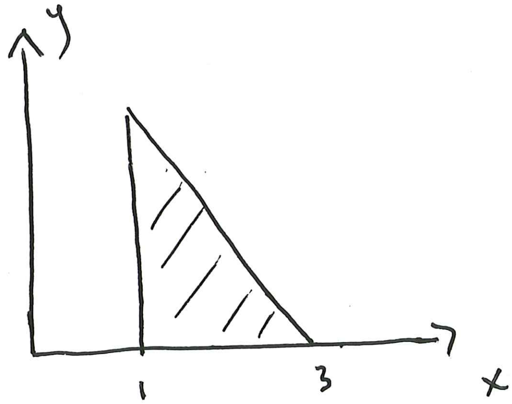
Choose first ~~integ~~ variable ...

For me, z is the easiest [redacted] choice:



$$\underbrace{\iint}_{} \int_{z=0}^{z=\sqrt{x}} f(x,y,z)\, dz \ \underbrace{d\_\, d\_}_{}$$

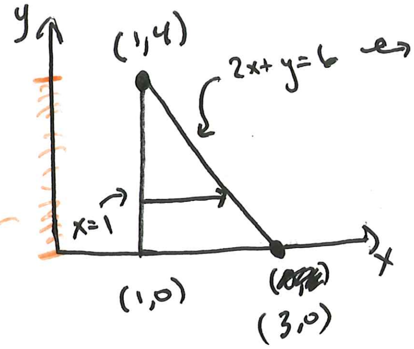
To pick the next, think about the [redacted] points (x,y) on which integrating z from $z=0$ to $z=\sqrt{x}$ makes sense!



With this domain, integrating either x or y next works without too much complication. We'll do it both ways!

If we do x next:

Note: w.r.t. = with respect to.

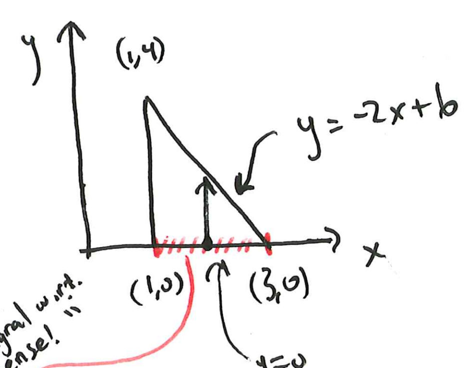


"set of y over which the int. w.r.t. x makes sense!"

1

$$\boxed{\int_{y=0}^{y=4}\int_{x=1}^{x=\frac{1}{2}(6-y)}\int_{z=0}^{z=\sqrt{x}} f(x,y,z)\,dz\,dx\,dy} \qquad (*)$$

If, instead, we do y next:

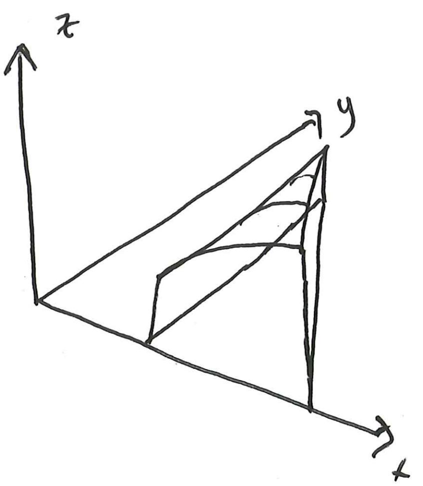


"set of x over which the integral w.r.t. y makes sense!"

2

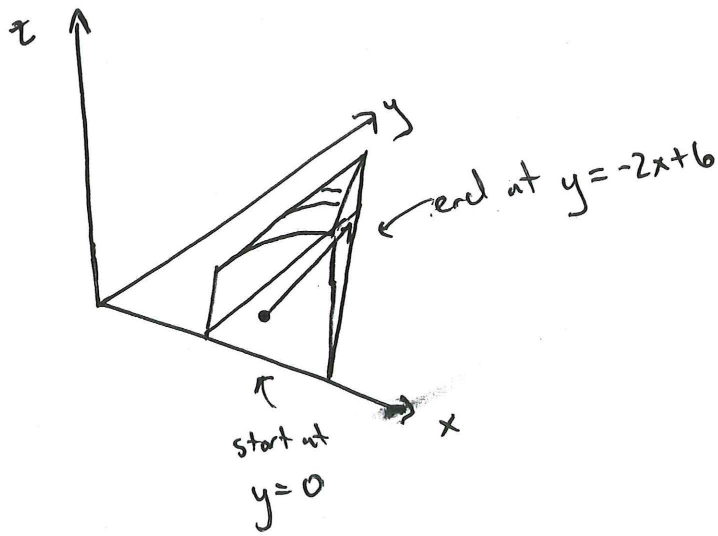
$$\boxed{\int_{x=1}^{x=3}\int_{y=0}^{y=-2x+6}\int_{z=0}^{z=\sqrt{x}} f(x,y,z)\,dz\,dy\,dx} \qquad (*)$$

To find a third way, we need to go back to the original domain and re-pick our first variable of integration.



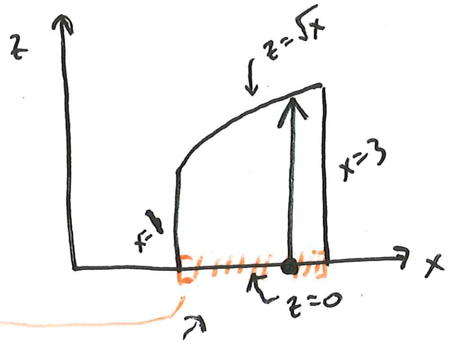
Notice, if we try to integrate w.r.t. x first, we run into a problem! I'll come back to this ...

Let's try integrating y first:



$$\underbrace{\int\int} \int_{y=0}^{y=-2x+6} f(x,y,z)\,dy \;\underbrace{d\_\,d\_}$$

Then ask "What values of (x,z) are valid for this integration scheme on y?". (Project onto xz-plane

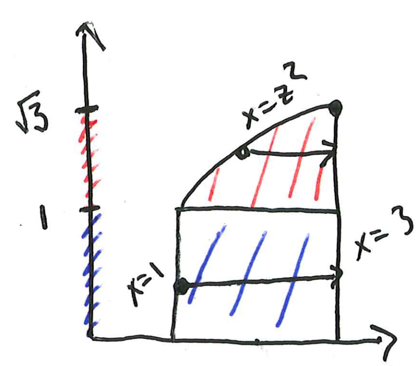


If we integrate z next, we will get one integral. Let's do that:

$$\boxed{\int_{x=1}^{x=3}\int_{z=0}^{z=\sqrt{x}}\int_{y=0}^{y=-2x+6} f(x,y,z)\,dy\,dz\,dx}$$ (3) . (✗) .

That finishes the problem... but I'll show the other three possible solutions.

First, integrate y first, like above, but now integrate x second.
(Not the best idea...)

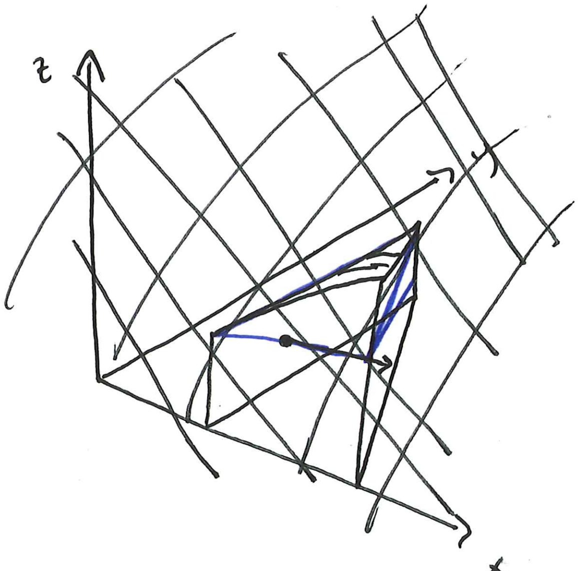


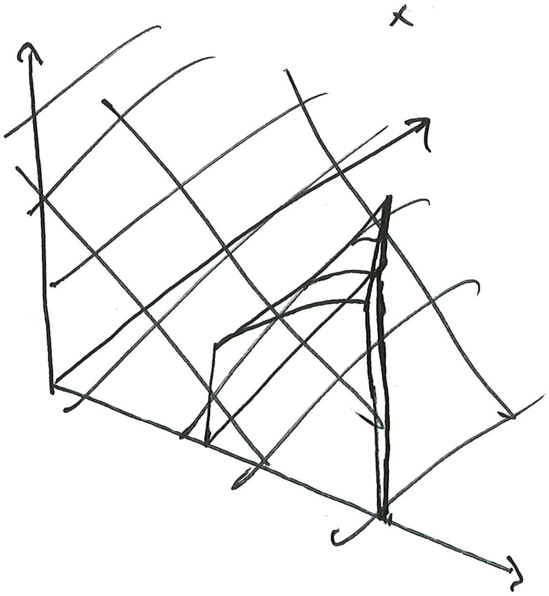
↙ we now need $\underline{\underline{2}}$ integrals! one where x goes from 1 to 3, and another where x goes from z^2 to 3.

$$\int_{z=0}^{1}\int_{x=1}^{x=3}\int_{y=0}^{y=-2x+6} f(x,y,z)\,dy\,dx\,dz + \int_{z=1}^{z=\sqrt{3}}\int_{x=z^2}^{x=3}\int_{y=0}^{y=-2x+6} f(x,y,z)\,dy\,dx\,dz$$ (4) (*)

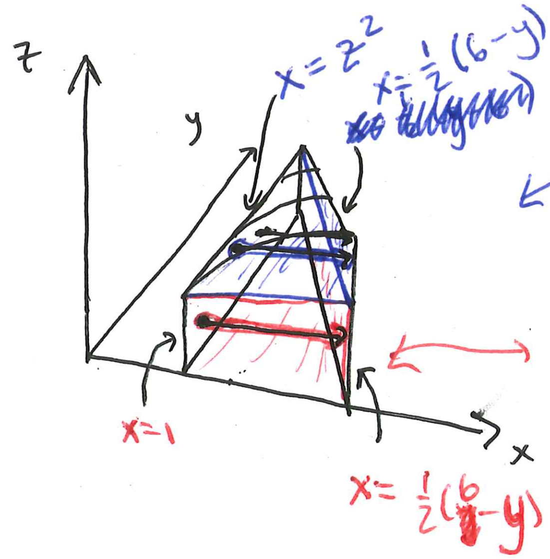
To get the other two, we need to go back to the first variable ~~[illegible]~~ of integration ... and make it x.

If you did this ... [redacted]it hit the fan. Not really, you just had to be careful.





I'll get it eventually.



top:

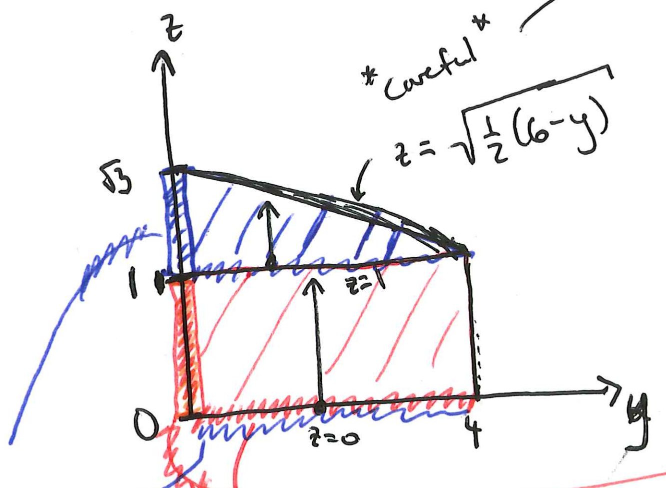
$$\iint \boxed{\int_{x=z^2}^{x=\frac{1}{2}(6-y)} f(x,y,z)\,dx} \;\_\_\_$$

$$+$$

bottom:

$$\iint \boxed{\int_{x=1}^{x=\frac{1}{2}(6-y)} f(x,y,z)\,dx} \;\_\_\_$$

Next, we need to project each region onto the yz-plane individually:



$$\begin{cases} 2x+y=6 \Rightarrow x = \frac{1}{2}(6-y) \\ z = \sqrt{x} = \sqrt{\frac{1}{2}(6-y)} \end{cases}$$

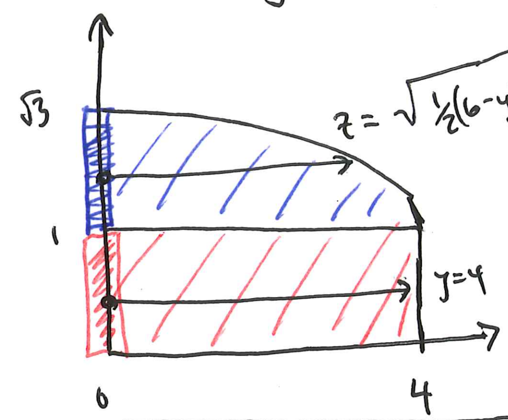
If we integrate z second:

$$\boxed{\boxed{\int_{y=0}^{y=4} \int_{z=1}^{z=\sqrt{\frac{1}{2}(6-y)}} \int_{x=z^2}^{x=\frac{1}{2}(6-y)} f(x,y,z)\,dx\,dz\,dy} + \boxed{\int_{y=0}^{y=4} \int_{z=0}^{z=1} \int_{x=1}^{x=\frac{1}{2}(6-y)} f(x,y,z)\,dx\,dz\,dy}}$$

5

(*)

Next, if we integrate y second:



$z=\sqrt{\tfrac{1}{2}(6-y)} \Leftrightarrow \underline{y=6-2z^2}$

(6)

$$\boxed{\int_{z=1}^{z=\sqrt{3}}\int_{y=0}^{y=6-2z^2}\int_{x=z^2}^{x=\frac{1}{2}(6-y)} f(x,y,z)\,dx\,dy\,dz} + \boxed{\int_{z=0}^{z=1}\int_{y=0}^{y=4}\int_{x=1}^{x=\frac{1}{2}(6-y)} f(x,y,z)\,dx\,dy\,dz} \qquad (*)$$

<u>REMARK</u> Definitely <u>don't</u> integrate w.r.t. x first. ☹