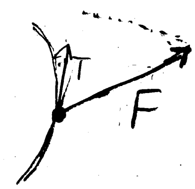
16.9 Divergence Theorem (Gauss's Theorem)

REM Green's Theorem

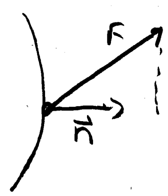
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \operatorname{curl}(\vec{F})\, dA.$$

In lecture notes for 16.5, we recalled that this line integral is $\oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot\vec{T}\, ds$, where $\vec{T}$ is the unit tangent vector.



"$\vec{F}\cdot\vec{T}$ is the amount of $\vec{F}$ in the direction of $\vec{T}$."

We then asked if we could glean any interesting information about the vector field by using an outward normal vector in lieu of the unit tangent vector.

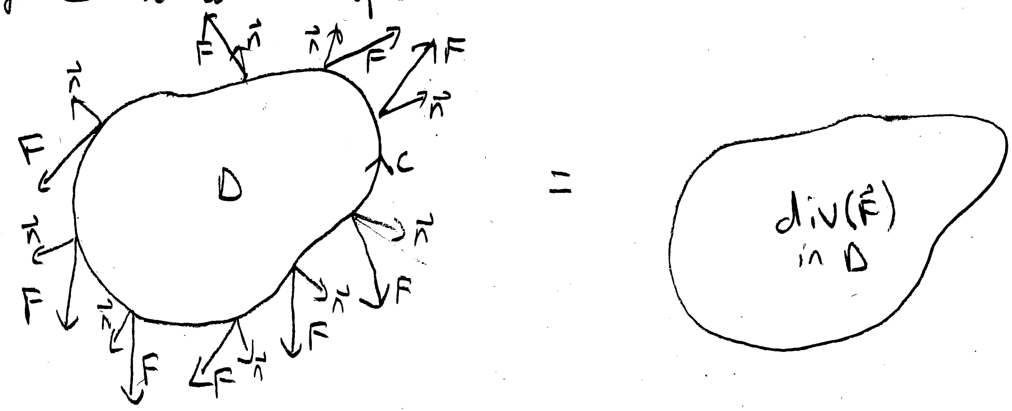


"$\vec{F}\cdot\vec{n}$ is the amount of $\vec{F}$ in the direction of $\vec{n}$."

Turns out it does: we can derive the following

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \iint_D \operatorname{div}(\vec{F})\, dA.$$

To see this computation, see the beginning of the Lecture #15 notes. It turns out that this is a 2-dimensional version of the Divergence theorem. It says that the amount of of flow (or flux) leaving a region is the sum of the divergence across the space:



This statement holds in higher dimensions! We state it for dim-3 below:

THM (Divergence Theorem)

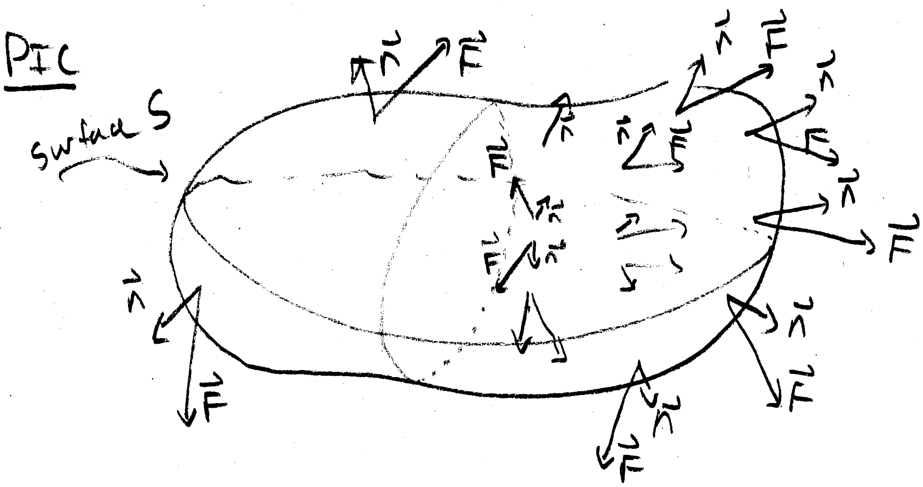
Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\boxed{\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_E \operatorname{div}(\vec{F})\, dV} \quad (*)$$

remark: Book may write

$$\iint_S \vec{F}\cdot\vec{n}\, dS \text{ as } \iint_S \vec{F}\cdot d\vec{S}. \quad \left(\text{Here } d\vec{S} = \vec{n}\, dS\right)$$

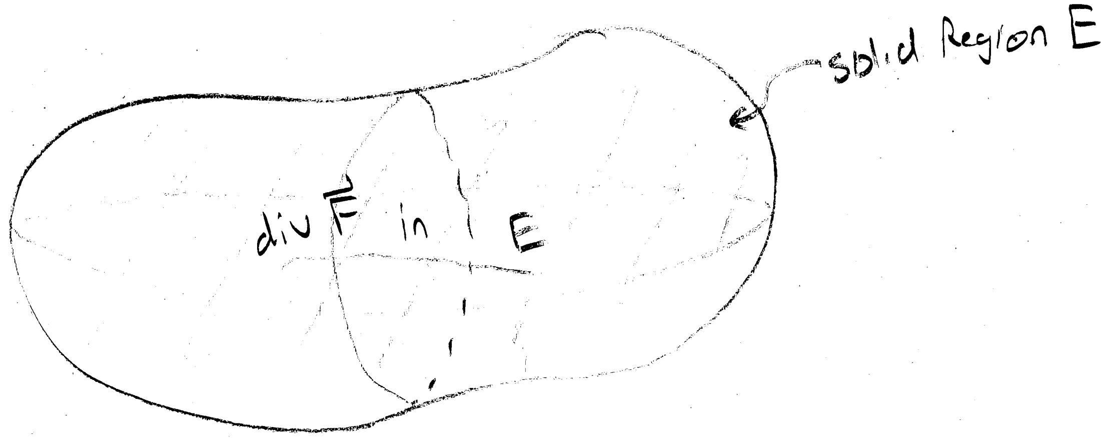
PIC



"How much is leaving"

" Flux through the surface S "

This is equal to :



REM Divergence can be thought of in terms of fluid dynamics; the divergence at a point is how much the fluid "diverges" from this point. If the fluid "diverges" at every point in a region(E), it must leave the region!, i.e. pass through the surface(S) of the region.

Proof of Divergence Theorem

Let $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ and let it have continuous partial derivatives on a $\overset{\text{open}}{\wedge}$ region containing E.

rem $\quad \operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

so $$\iiint_E \operatorname{div}(\vec{F})\,dV = \iiint_E \frac{\partial P}{\partial x}\,dV + \iiint_E \frac{\partial Q}{\partial y}\,dV + \iiint_E \frac{\partial R}{\partial z}\,dV.$$

Now, let $\vec{n}$ denote the outward normal on S, the boundary of E, and we see

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint (P\hat{\imath} + Q\hat{\jmath} + R\hat{k})\cdot\vec{n}\,dS$$

notice: $P\hat{\imath}\cdot\vec{n} = \langle P,0,0\rangle\cdot\langle n_1,n_2,n_3\rangle = Pn_1$ etc.

$$= \iint_S P\hat{\imath}\cdot\vec{n}\,dS + \iint_S Q\hat{\jmath}\cdot\vec{n}\,dS + \iint_S R\hat{k}\cdot\hat{n}\,dS.$$

We show that

$$\iiint_E \frac{\partial P}{\partial x}\,dV = \iint_S P\hat{\imath}\cdot\hat{n}\,dS,$$ (and similarly for the other two terms!)

To do this, we will make one additional assumption (the proof becomes much harder if you do not do this!). We need to assume that E is a nice region, i.e. can be written as $E := \{(x,y,z) : (y,z)\in D \text{ and } u_1(y,z) \le x \le u_2(y,z)\}$, where $D := \{(y,z) : a \le z \le b,\ f_1(z) \le y \le f_2(z)\}$ or $\{(y,z) : a \le y \le b,\ f_1(y) \le z \le f_2(y)\}$.

We know from our work with triple integrals that this isn't always possible!

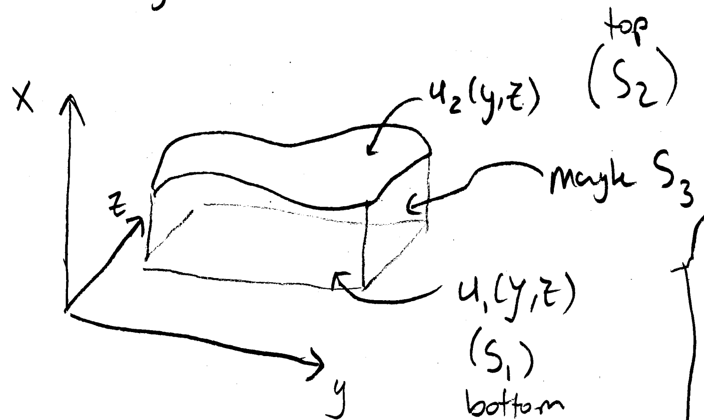
If this is true, then

$$\iiint_E \frac{\partial P}{\partial x}\, dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} \frac{\partial P}{\partial x}\, dx \right) dA$$

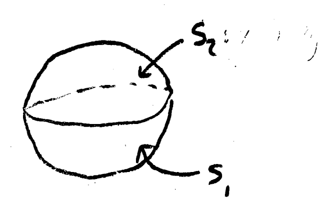
Fundamental Theorem of Calculus!

$$= \iint_D \left[P(u_2(y,z), y, z) - P(u_1(y,z), y, z) \right] dA.$$

Next, consider $\iint_S P\hat{\imath} \cdot \vec{n}\, dS$. The surface will have a top and bottom, and ~maybe~ a side:



(Notice, a sphere has no S_3!



$S_2: x = \sqrt{1-y^2-z^2}$

$S_1: x = -\sqrt{1-y^2-z^2}$

So break S into S_1, S_2, S_3:

$$\iint_S P\hat{\imath} \cdot \vec{n}\, dS = \iint_{S_1} P\hat{\imath} \cdot \vec{n}\, dS + \iint_{S_2} P\hat{\imath} \cdot \vec{n}\, dS + \iint_{S_3} P\hat{\imath} \cdot \vec{n}\, dS.$$

If there is an S_3, notice, because of how our region E is defined

$$\hat{\imath} \cdot \vec{n} = 0$$

since $\hat{\imath}$ points in the x-direction (up in the picture) and $\vec{n}$ must be parallel to the yz-plane. In other words, they are normal vectors, so the dot product is 0.

This means

$$\iint_{S_3} P\hat{\imath} \cdot \vec{n} \, dS = \iint_{S_3} 0 \, dS = 0.$$

Thus,

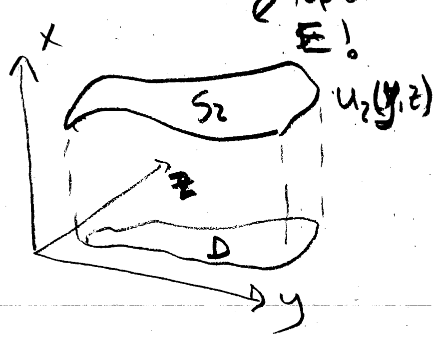
$$\iint_{S} P\hat{\imath} \cdot \vec{n} \, dS = \iint_{S_1} P\hat{\imath} \cdot \vec{n} \, dS + \iint_{S_2} P\hat{\imath} \cdot \vec{n} \, dS.$$

Consider

$$\iint_{S_2} P\hat{\imath} \cdot \vec{n} \, dS = \iint_{D} \langle P, 0, 0 \rangle \cdot \langle 1, \frac{-\partial u_2}{\partial y}, \frac{-\partial u_2}{\partial z} \rangle \, dA$$

this is the parameter domain, same D as before!

top of E!



$$\vec{n} = \frac{\vec{r}_y \times \vec{r}_z}{|\vec{r}_y \times \vec{r}_z|}$$

(graph of a function!)

$$= \frac{1}{|\vec{r}_y \times \vec{r}_z|} \cdot \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial u_2}{\partial y} & 1 & 0 \\ \frac{\partial u_2}{\partial z} & 0 & 1 \end{vmatrix} = 1\,\hat{\imath} + \frac{-\partial u_2}{\partial y}\hat{\jmath} + \frac{-\partial u_2}{\partial z}\hat{k}$$

So,

$$\iint_{S_2} P\hat{\imath} \cdot \vec{n}\, dS = \iint_D P\, dA$$

evaluated at $\begin{cases} x = u_2(y,z) \\ y = y \\ z = z \end{cases}$

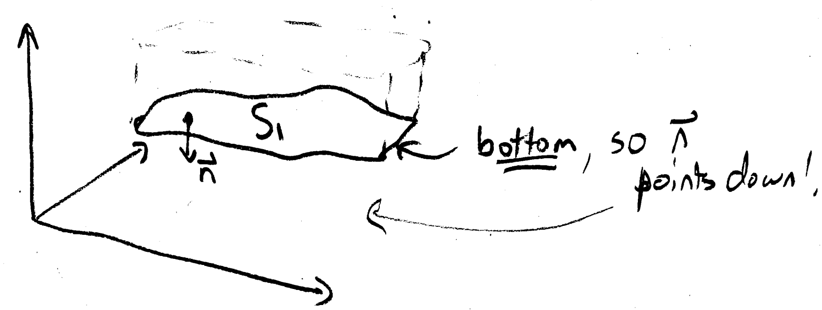
$$= \iint_D P(u_2(y,z), y, z)\, dA.$$

Similarly

$$\iint_{S_1} P\hat{\imath} \cdot \vec{n}\, dS = -\iint_D P\, dA = -\iint_D P(u_1(y,z), y, z)\, dA$$

$$\vec{n} = -\frac{\vec{r}_y \times \vec{r}_z}{|\vec{r}_y \times \vec{r}_z|}$$

* we need to flip the normal vector!



Thus,

$$\iint_S P\hat{\imath} \cdot \vec{n}\, dS = \iint_D \Big(P(u_2(y,z), y, z) - P(u_1(y,z), y, z) \Big)\, dA$$

which matches our computation for $\iiint_E \frac{\partial P}{\partial x}\, dV$.

The other two terms work similarly! $\blacksquare$

EXAMPLE 1

Find the flux of the vector field $\vec{F} = z^3 y\,\hat{\imath} + y\,\hat{\jmath} + \sin(x)\,\hat{k}$ over the unit sphere $x^2+y^2+z^2 = 1$.

We are being asked to compute $\iint_S \vec{F}\cdot\vec{n}\,dS$, the flux. Notice that

- S is the boundary of a simple solid region $E := \{(x,y,z) : x^2+y^2+z^2 \le 1\}$.
- $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}, \frac{\partial R}{\partial z}$ are continuous in E (and actually all of $\mathbb{R}^3$).

$\Rightarrow$ We can apply the divergence theorem.

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_E \operatorname{div}(\vec{F})\,dV$$

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + 1 + 0 = 1,$$

$\frac{4\pi r^3}{3}$

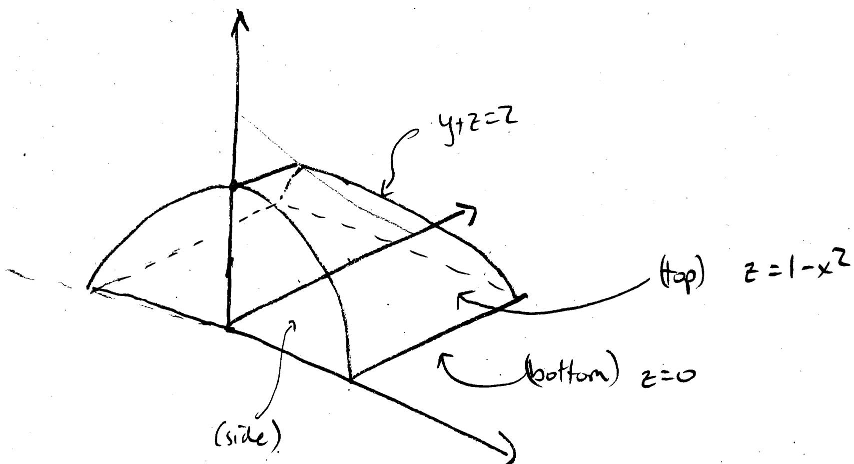
Thus $$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_E 1\,dV = \iiint dV = V(\text{unit sphere}) = \frac{4\pi(1)^3}{3}$$

$$\boxed{= \frac{4\pi}{3}}$$

EXAMPLE 2 Evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where

$$\vec{F}(x,y,z) = xy\,\hat{\imath} + (y^2 + e^{xz^2})\,\hat{\jmath} + \sin(xy)\,k$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z=0$, $y=0$, and $y+z=2$.



- S is the boundary of a simple solid region
-

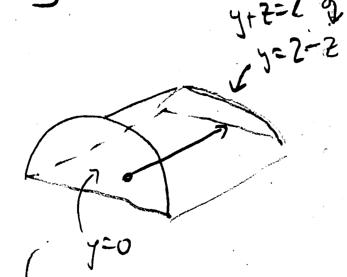
Use Divergence Theorem!

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\,dV$$

$$\operatorname{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = y + 2y + 0 = 3y$$

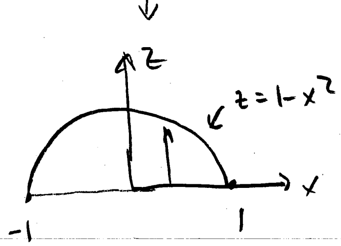
$$= \iiint_E 3y\,dV$$

integrate w.r.t y first



$$= \iint \int_0^{2-z} 3y\,dV$$

integrate z next



$$\boxed{= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y\,dy\,dz\,dx}$$

$$= 3\int_{-1}^{1}\int_{0}^{1-x^2} \left[\frac{y^2}{2}\right]_0^{2-z} dz\,dx$$

$$= \frac{3}{2}\int_{-1}^{1}\int_{0}^{1-x^2} (2-z)^2\, dz\,dx$$

$u = 2-z$

$du = -1\cdot dz$

$$= \frac{3}{2}\int_{-1}^{1}\left[-\frac{(2-z)^3}{3}\right]_0^{1-x^2} dx$$

$$= \frac{1}{2}\int_{-1}^{1}\left(-(2-(1-x^2))^3 + 8\right)$$

$$= \frac{-1}{2}\int_{-1}^{1}\left(((1+x^2)^3 - 8\right) dx$$

$$= \frac{-1}{2}\int_{-1}^{1}\left[(1+x^2)(1+2x^2+x^4) - 8\right] dx$$

$$= \frac{-1}{2}\int_{-1}^{1}\left[1+2x^2+x^4+x^2+2x^4+x^6 - 8\right] dx$$

$$= \frac{-1}{2}\int_{-1}^{1}\left(3x^2+3x^4+x^6-7\right) dx$$

$$= \frac{-1}{2}\left[x^3 + \frac{3x^5}{5} + \frac{x^7}{7} - 7x\right]_{-1}^{1}$$

$$= \frac{-1}{2}\left[\left(1+\frac{3}{5}+\frac{1}{7}-7\right) - \left(-1+\frac{-3}{5}+\frac{-1}{7}+7\right)\right]$$

$$= \frac{-1}{2}\left[2+\frac{6}{5}+\frac{2}{7}-14\right] = \frac{-1}{2}\left[\frac{70}{35}+\frac{42}{35}+\frac{10}{35}-\frac{490}{35}\right]$$

$$\boxed{= \frac{184}{35}}$$