Lecture #19

16.8 Stokes' Theorem

REM Green's Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

In 16.5, we noticed that we could rewrite this expression using the curl operator, provided we think of $\vec{F}$ as a "3-dim" vector field, in other words

$$\vec{F} = P\hat{\imath} + Q\hat{\jmath} \overset{\text{"think of as"}}{\Longrightarrow} \vec{F} = P\hat{\imath} + Q\hat{\jmath} + \underbrace{0}_{\text{0 in the } \hat{k} \text{ direction!}}\hat{k}.$$

The expression reduced to

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \operatorname{curl}(\vec{F})\cdot\hat{k}\ dA.$$

Notice that the vector field $\vec{F}$ is constrained to move only in the xy-plane (and planes parallel to the xy-plane). This means $\hat{k}$ is the direction normal to the movement in the vector field.

This is actually true in a more general setting:

<u>THM</u> (Stokes' Theorem)

Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \iint_S \operatorname{curl}(\vec{F}) \cdot \vec{n}\, dS .$$

<u>EXERCISE</u> Compare this to Green's Theorem. Derive Green's Theorem from Stokes' theorem. (What do C and S need to be? What does the vector field need to be?)

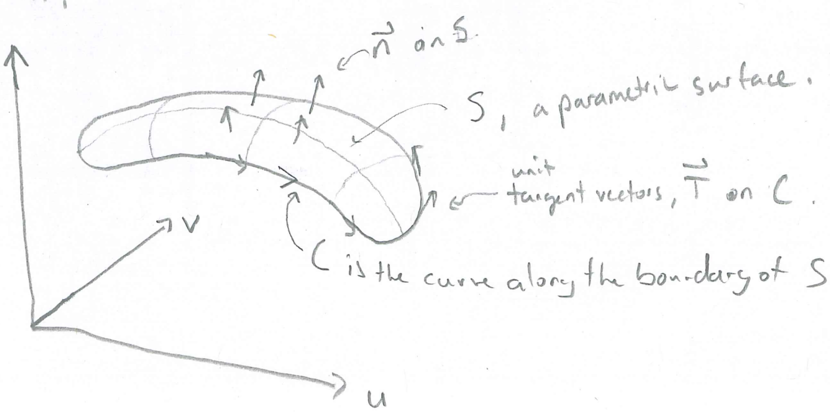
$\Rightarrow$ C needs to be a simple, closed, piecewise smooth curve <u>in $\mathbb{R}^2$</u>.

$\Rightarrow$ S needs to be the region bounded by C in $\mathbb{R}^2$

$\Rightarrow$ $\vec{F}$ needs to have no $\hat{k}$ component

$\Rightarrow$ This shows we can think of Green's Theorem as a consequence of Stokes' Theorem, or equivalently, as Stokes' Theorem being a generalization of Green's Theorem!

<u>PIC</u>

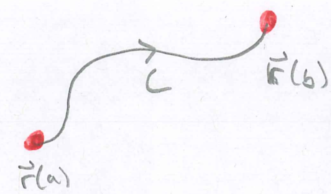


(*)

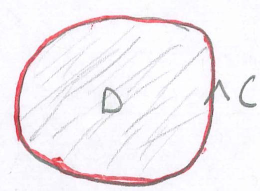
REMARK The key idea to notice is that Stoke's Theorem is relating an integral over a region to an integral over a boundary. Just like Green's Theorem, we need to recognize Stokes' Theorem as a generalization of the Fundamental theorem of Calculus:

FTC $\quad \displaystyle\int_a^b F'(x)\,dx = F(\underline{b}) - F(\underline{a})$

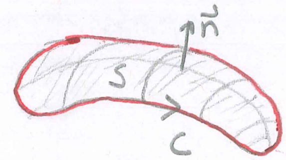
Fundamental Theorem of Line Integrals $\quad \displaystyle\int_C \nabla f \cdot d\vec{r} = f(\underline{\vec{r}(b)}) - f(\underline{\vec{r}(a)})$



Green's Theorem $\quad \displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\underline{C}} P\,dx + Q\,dy$

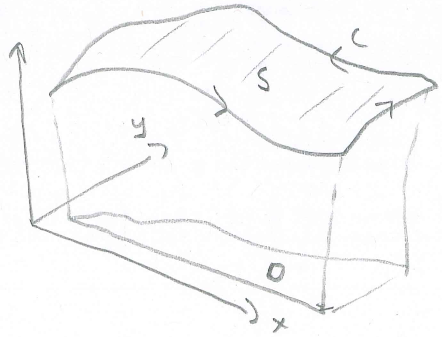


Stokes' Theorem $\quad \displaystyle\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \oint_{\underline{C}} \vec{F} \cdot d\vec{r}$



<u>Proof of Stokes' Theorem</u> (special case: S is the <u>graph of a function</u>)

Assume S is the graph of a function, so $\vec{r}(x,y) = x\hat{\imath} + y\hat{\jmath} + z(x,y)\hat{k}$ and $(x,y) \in D$:



Also, assume $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, where partial derivatives of P, Q, R are continuous.

We can compute:

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_S \underbrace{\text{curl}(\vec{F})} \cdot \vec{n}\, dS$$

$$= \iint_D \text{curl}(\vec{F}(\vec{r}(x,y))) \cdot \vec{n} \cdot |\vec{r}_x \times \vec{r}_y|\, dA$$

$$= \iint_D \text{curl}(\vec{F}(\vec{r}(x,y))) \cdot \frac{\vec{r}_x \times \vec{r}_y}{\cancel{|\vec{r}_x \times \vec{r}_y|}} \cancel{|\vec{r}_x \times \vec{r}_y|}\, dA$$

$$= \iint_D \text{curl}(\vec{F}(\vec{r}(x,y))) \cdot (\vec{r}_x \times \vec{r}_y)\, dA$$

$$\text{curl}(\vec{F}(\vec{r}(x,y))) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

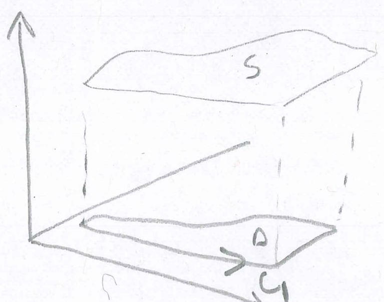
$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ 1 & 0 & \frac{\partial y}{\partial x} \\ 0 & 1 & \frac{\partial y}{\partial y} \end{vmatrix} = \frac{-\partial z}{\partial x}\hat{\imath} + \frac{-\partial z}{\partial y}\hat{\jmath} + \hat{k}$$

$(\vec{r} = x\hat{\imath} + y\hat{\jmath} + z(x,y)\hat{k})$

Then $$\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S} = \iint_D \left\langle \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} \right\rangle \cdot \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle dA$$

$$= \iint_D \left[-\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x} + -\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\right] dA$$

Next, we need to know about the boundary. Assume we can parametrize the boundary of D as follows (call it C_1)

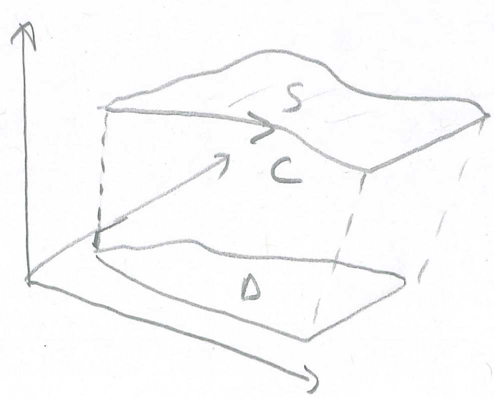


boundary of D: C_1

$$\begin{cases} x = x(t) \\ y = y(t), \quad a \le t \le b \end{cases}$$

Then the boundary of S, our curve C, can be parametrized by

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(x(t), y(t)), \quad a \le t \le b \end{cases} \Rightarrow \vec{r}_2(t) = \langle x(t), y(t), z(x(t), y(t)) \rangle$$



Then, we can compute:

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{t=a}^{t=b} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2\,'(t)\,dt$$

functions of t

$\frac{dz}{dt}$ rem $z(x(t), y(t))$ chain rule!

$$= \int_a^b \langle P, Q, R\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\right\rangle dt$$

$$= \int_a^b P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\right) dt$$

$$= \int_a^b \left(\left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right) dt$$

"un" parametrize. (look at this step backwards! $x(t) \to dx = x'(t)dt$, $x'(t) = \frac{dx}{dt}$)

$$= \int_{C_1} \left(P + R\frac{\partial z}{\partial x}\right)dx + \left(Q + R\frac{\partial z}{\partial y}\right)dy$$

this is C_1, not C! (why?)

why did we do that? To apply Green's Theorem!

$$= \iint_D \left(\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right) dA$$

chain rule! rem that $Q(x,y,z) = Q(x,y,z(x,y))$ (same for P, R!)

$$= \iint_D \Bigg[\underbrace{\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\cdot\frac{\partial z}{\partial x}\right)}_{\frac{\partial}{\partial x}(Q)} + \underbrace{\frac{\partial R}{\partial x}\cdot\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\cdot\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}}_{\frac{\partial}{\partial x}(R)\cdot\frac{\partial z}{\partial y}} + \underbrace{R\cdot\frac{\partial^2 z}{\partial x \partial y}}_{R\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)}\Bigg)$$

lots of cancellation!

$$- \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\cdot\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\cdot\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\cdot\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y \partial x}\right)\Bigg] dA$$

$$= \iint_D \left[\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) + -\left(\frac{\partial R}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + -\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} \right] dA$$

matches our computation for $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$!

$\Rightarrow$ This proves Stokes' Theorem for graphs of functions!

EXAMPLE 1 Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = -y^2 \hat{\imath} + x \hat{\jmath} + z^2 \hat{k}$ and C is the curve of the intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$. (Orient C to be counterclockwise when viewed from above.

$$\oint_C F \cdot d\vec{r} = \ \ldots$$

use Stokes' Theorem

- Notice, first order partials of P, Q, R are defined and continuous in all of $\mathbb{R}^3$
- The curve C has the correct orientation, for an "upward" normal vector.

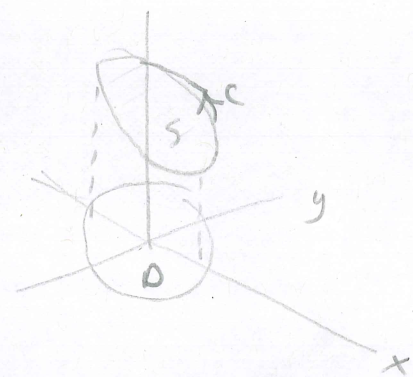
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot \vec{n} \, dS$$

STEP 1 Compute the curl

$$\text{Curl}(\vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + (1+2y)\hat{k}$$

$$= (1+2y)\hat{k}$$

STEP 2 Parametrize S (Notice it is the graph of a function!)



$$S: \quad \begin{aligned} x &= x \\ y &= y \\ z &= 2-y \end{aligned}$$

$$\vec{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + (2-y)\hat{k}$$

$$D = \{(x,y) : x^2+y^2 \le 1\}$$

STEP 3 Notice:

$$\iint_S \text{Curl}(\vec{F})\cdot\vec{n}\,dS = \iint_D \text{curl}(\vec{F})\cdot\frac{(\vec{r}_x\times\vec{r}_y)}{\cancel{|\vec{r}_x\times\vec{r}_y|}}\cdot\cancel{|\vec{r}_x\times\vec{r}_y|}\,dA$$

and $\vec{r}_x \times \vec{r}_y = -\frac{\partial z}{\partial x}\hat{\imath} + -\frac{\partial z}{\partial y}\hat{\jmath} + \hat{k}$ (this was an exercise..)

$$= 0\,\hat{\imath} + -1\,\hat{\jmath} + 1\,\hat{k}$$

STEP 4 Compute!

$$\iint_S \text{curl}(\vec{F})\cdot d\vec{S} = \iint_D \text{curl}(\vec{F})\cdot(\vec{r}_x\times\vec{r}_y)\,dA$$

$$= \iint_D (1+2y)\, dA$$

D is a disk!
use polar coordinates!

$$= \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r + 2r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + \frac{2r^3}{3}\sin\theta\right]_0^1 d\theta$$

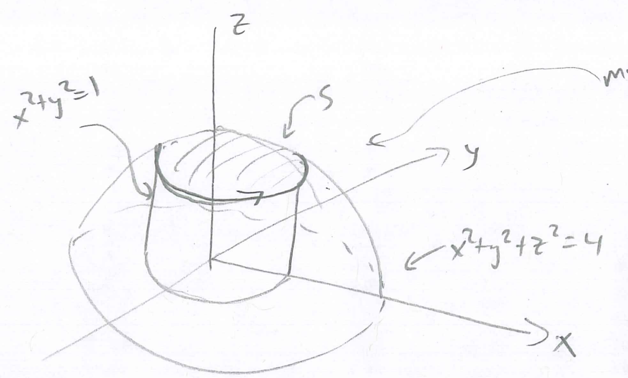
$$= \int_0^{2\pi} \left[\frac{1}{2} + \frac{2}{3}\sin\theta\right] d\theta$$

$$= \left[\frac{1}{2}\theta - \frac{2}{3}\cos\theta\right]_0^{2\pi}$$

$$= \left(\pi - \frac{2}{3}\right) - \left(0 - \frac{2}{3}\right)$$

$$\boxed{= \pi}$$

<u>EXAMPLE 2</u> Use Stokes' Theorem to compute $\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}$, where $\vec{F}(x,y,z) = xz\,\hat{\imath} + yz\,\hat{\jmath} + xy\,\hat{k}$ and S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane.



make the boundary C!

Stokes' Theorem!

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

STEP 1 Parametrize the boundary:

Find the intersection of the cylinder and the sphere

$$\begin{cases} x^2+y^2=1 \\ x^2+y^2+z^2=4 \end{cases} \Rightarrow \begin{aligned} 1+z^2 &= 4 \\ z^2 &= 3 \\ z &= \pm\sqrt{3} \end{aligned}$$

We are looking for positive z, so $\boxed{z=\sqrt{3}}$.

Then, our parametrization follow $x^2+y^2=1$ at a height of $\sqrt{3}$:

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = \sqrt{3} \end{cases}, \quad 0 \le t \le 2\pi$$

So, $\vec{r}(t) = \cos t\, \hat{\imath} + \sin t\, \hat{\jmath} + \sqrt{3}\, \hat{k}$

and $\vec{r}\,'(t) = -\sin t\, \hat{\imath} + \cos t\, \hat{\jmath} + 0\, \hat{k}$

STEP 2 Plug parametrization: $\vec{F}(\vec{r}(t)) = (\cos t)\sqrt{3}\, \hat{\imath} + (\sin t)\sqrt{3}\, \hat{\jmath} + \cos t \sin t\, \hat{k}$

STEP 3 Compute!

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{t=2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \cos t\sin t\rangle \cdot \langle -\sin t, \cos t, 0\rangle\,dt$$

$$= \int_0^{2\pi} \left(-\sqrt{3}\cos t\sin t + \sqrt{3}\sin t\cos t\right)dt$$

$$= \int_0^{2\pi} 0\,dt$$

$$= 0$$

<u>QUESTION</u> If S is a sphere (eg. $x^2+y^2+z^2=1$) and $\vec{F}$ is a vector field that satisfies the conditions of Stokes' Theorem, what is $\iint_S \operatorname{curl}(\vec{F})\cdot\vec{n}\,dS$?

<u>EXERCISES</u> #1; #2,5; #7,9; #15, #16, #18-20. (16.8)