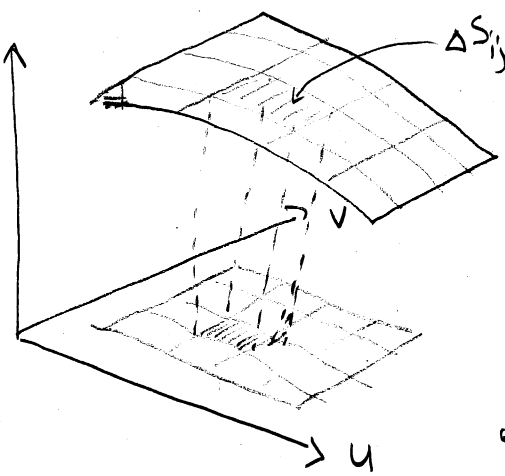
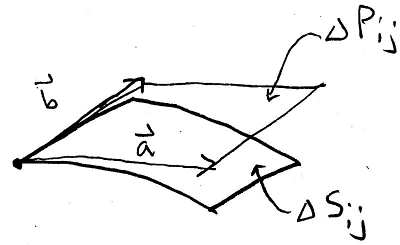
16.6 (cont.)

Recall from 15.5 (text) : Surface integrals



we will think of this as a "uv"-plane when working with $\vec{r}(u,v)$, i.e. parametric surfaces.

Computing a small piece of Surface area:



We estimate small pieces by the area of a parallelogram!

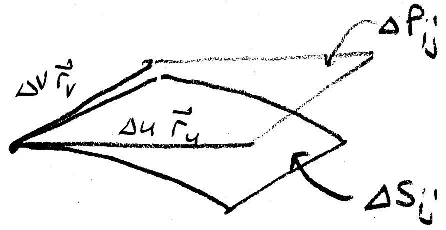
To do this, we took the magnitude of the cross product of $|\vec{a} \times \vec{b}|$, since this gives us the area of the pallelogram spanned by $\vec{a}$ and $\vec{b}$.

From last lecture, recall

$\underline{\vec{r}_u}$ and $\underline{\vec{r}_v}$ ("partials in the direction of u and v on a parametric surface")

We can use these in the place of $\vec{a}$ and $\vec{b}$!

If you look back to the lecture notes on Surface area, you will see that this is secretly what we were using all along, but instead of u,v, we used x and y, and constrained ourselves to usual surfaces (not parametric ones!) So,



Computing:

$$\text{Area}(\Delta S_{ij}) \approx \text{Area}(\Delta P_{ij}) = |\Delta u \vec{r}_u \times \Delta v \vec{r}_v| = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v.$$

Summing over each pieces in the surface:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |r^*_{u,ij} \times r^*_{v,ij}| \Delta u \Delta v$$

and taking a limit, we get:

<u>DEF</u> If a smooth parametric surface S is given by the equation

$$\vec{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}, \quad (u,v) \in D$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the <u>surface area</u> of S is

$$\boxed{A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA}$$

where $\vec{r}_u = \frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath} + \frac{\partial z}{\partial u}\hat{k}$ and

$\vec{r}_v = \frac{\partial x}{\partial v}\hat{\imath} + \frac{\partial y}{\partial v}\hat{\jmath} + \frac{\partial z}{\partial v}\hat{k}$

EXAMPLE 9 Find the surface area of a sphere of radius a.

rem: Parametric equation of a sphere:

$$x = a\sin\phi\cos\theta$$
$$y = a\sin\phi\sin\theta$$
$$z = a\cos\phi,$$

[For $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.]

This is the parameter domain $\underline{D}$;

$$D = \{(\phi,\theta) : 0 \le \phi \le \pi, 0 \le \theta \le 2\pi\}$$

We start by computing $\vec{r}_\phi \times \vec{r}_\theta$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\cos\theta\,\hat{\imath} + a^2\sin^2\phi\sin\theta\,\hat{\jmath} + (a^2\cos\phi\cos^2\theta\sin\phi + a^2\cos\phi\sin\phi\sin^2\theta)\hat{k}$$

$$= a^2\sin^2\phi\cos\theta\,\hat{\imath} + a^2\sin^2\phi\sin\theta\,\hat{\jmath} + (a^2\cos\phi\sin\phi)k$$

Then: $$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\cos^2\phi\sin^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\cos^2\phi\sin^2\phi}$$

$$= a^2\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}$$

$$= a^2\sqrt{\sin^4\phi + (1-\sin^2\phi)(\sin^2\phi)}$$

$$= a^2\sqrt{\sin^4\phi + \sin^2\phi - \sin^4\phi}$$

$$= a^2\sqrt{\sin^2\phi} = \boxed{a^2\sin\phi}$$

* what does this remind you of? (Jacobian of sorts...?)

Then $\iint_D |\vec{r}_\phi \times \vec{r}_\theta| \, dA = \int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi \, d\phi \, d\theta$

$$= a^2 \int_0^{2\pi} -\cos\phi \Big]_0^{\pi} \, d\theta$$

$$= a^2 \int_0^{2\pi} 2 \, d\theta$$

$$= a^2 \left[2\theta\right]_0^{2\pi}$$

$$= a^2 \cdot 4\pi \quad \rightsquigarrow \quad \boxed{= 4\pi a^2}$$

EXERCISE Let your parametric surface be described by

$$\begin{cases} x = x \\ y = y \\ z = f(x,y) \end{cases}$$

Show $\boxed{A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA.}$

REMARK This is telling you that the graph of a function is a special case of a parametric surface, and our new equation for surface area agrees with our old one!

If you want to brush up on using this formula

EXERCISE Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

16.7 Surface Integrals

REMARK Compare what follows to our treatment of line integrals in 16.2. You'll find it analogous in many respects!

DEF We define a surface integral of f over the surface S as

just add the function in!

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(P_{ij}^*)\,\Delta S_{ij} = \lim_{m,n\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(P_{ij}^*)\,|\vec{r}_u\times\vec{r}_v|\,\Delta u\,\Delta v$$

Given a parametric equation $\vec{r}(u,v)$ describing the surface S, we can compute the surface integral as follows:

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u\times\vec{r}_v|\,dA.}$$

parameter domain

REMARK If $f(x,y,z)=1$, then $\iint_S f(x,y,z)\,dS = \iint_S dS = \iint_D |\vec{r}_u\times\vec{r}_v|\,dA = A(S)$

(surface) area of S.

EXAMPLE 1 Compute $\iint_S x^2\,dS$, where S is the unit sphere $x^2+y^2+z^2$.

STEP 1 Parametrize!

$$\vec{r}(\phi,\theta) = \langle \underset{x}{\sin\phi\cos\theta},\ \underset{y}{\sin\phi\sin\theta},\ \underset{z}{\cos\phi} \rangle$$

$$D = \{(\phi,\theta) : 0\le\phi\le\pi,\ 0\le\theta\le 2\pi\}$$

STEP 2 Compute $|\vec{r}_\phi\times\vec{r}_\theta|$. (We just did this! For radius a, we got $a^2\sin\phi$. Here, we get...)

$$\left|\vec{r}_\phi \times \vec{r}_\theta\right| = \sin\phi$$

<u>STEP 3</u>. Compute

$$\iint_S x^2\,dS = \iint_D (\underbrace{\sin\phi\cos\theta}_{x(\phi,\theta)})^2 \cdot \left|\vec{r}_\phi \times \vec{r}_\theta\right| dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi \cos^2\theta \; d\phi\, d\theta$$

$\sin^2\phi \cdot \sin\phi = (1-\cos^2\phi)\sin\phi$

$$= \int_0^{2\pi} \cos^2\theta \left(\int_0^{\pi} \left(\sin\phi - \underbrace{\cos^2\phi \sin\phi}_{u\text{-sub}}\right) d\phi\right) d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \cdot \left(\left(1-\tfrac{1}{3}\right) - \left(-1+\tfrac{1}{3}\right)\right) d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \cdot \left(\tfrac{4}{3}\right) d\theta$$

double angle identity!

$$= \frac{4}{3}\int_0^{2\pi} \frac{1}{2}\left(1 + \cos 2\theta\right) d\theta$$

$$= \frac{4}{3}\cdot\frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{4}{3}\cdot\frac{1}{2}\left[(2\pi + 0) - (0+0)\right]$$

$$= \frac{4}{3}\cdot\frac{1}{2}\cdot 2\pi$$

$$\boxed{= \frac{4}{3}\pi}$$

REMARK We can use surface integrals to find the "mass" of parametric surfaces with density $p(x,y,z)$:

$$m = \iint_S p(x,y,z)\,dS$$

and the "center of mass" $(\bar{x},\bar{y},\bar{z})$:

$$\bar{x} = \frac{1}{m}\iint_S x\,p(x,y,z)\,dS, \quad \bar{y} = \frac{1}{m}\iint_S y\,p(x,y,z)\,dS, \quad \bar{z} = \frac{1}{m}\iint_S z\,p(x,y,z)\,dS$$

Now, let's switch to a special case of parametric surfaces, graphs of functions (i.e. just a surface $z=g(x,y)$).

Rem we can always think of a graph of a function as

$$\begin{cases} x = x \\ y = y \\ z = g(x,y) \end{cases} \quad \text{"}\vec{r}(x,y)\text{"},$$

a set of parametric equations.

For this special case,

this is "$f(\vec{r}(x,y))$"

$$(*) \quad \boxed{\begin{aligned} \iint_S f(x,y,z)\,dS &= \iint_D f(x,y,g(x,y)) \cdot |\vec{r}_x \times \vec{r}_y|\,dA \\ &= \iint f(x,y,g(x,y)) \cdot \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA \end{aligned}} \quad (*)$$

EXAMPLE 2 Evaluate $\iint_S y\,dS$, where S is the surface

$z = x + y^2,\quad 0 \le x \le 1,\quad 0 \le y \le 2.$

STEP 1 : $\frac{\partial z}{\partial x} = 1$

$\frac{\partial z}{\partial y} = 2y$

STEP 2 : Use our new formula!

$$\iint_S y\,dS = \iint_D y \cdot \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \int_0^1 \int_0^2 y\sqrt{1 + 1^2 + 4y^2}\,dy\,dx$$

$$= \int_0^1 \int_0^2 y\sqrt{2 + 4y^2}\,dy\,dx$$

$u = 2 + 4y^2,\quad du = 8y\,dy$

$$= \int_0^1 \frac{1}{8}\int_{y=0}^{y=2} \sqrt{u}\,du\,dx$$

$$= \int_0^1 \frac{1}{8} \cdot \left[\frac{2}{3}(2+4y^2)^{3/2}\right]_0^2 dx$$

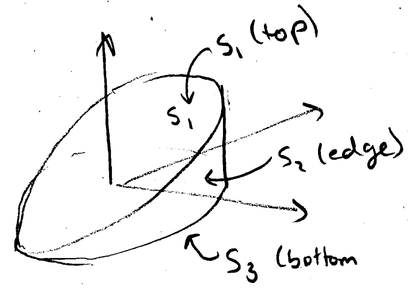
$$= \int_0^1 \frac{1}{8} \cdot \left[\frac{2}{3}\left((18\sqrt{18}) - 2\sqrt{2}\right)\right] dx$$

$$= \frac{1}{8} \cdot \frac{2}{3} \cdot (54\sqrt{2} - 2\sqrt{2})$$

$52 = 13 \cdot 4$

$$= \frac{1}{8} \cdot \frac{2}{3} \cdot 52\sqrt{2} = \boxed{\frac{13\sqrt{2}}{3}}$$

REMARK If a surface S is composed of a series of smaller surfaces, such as:

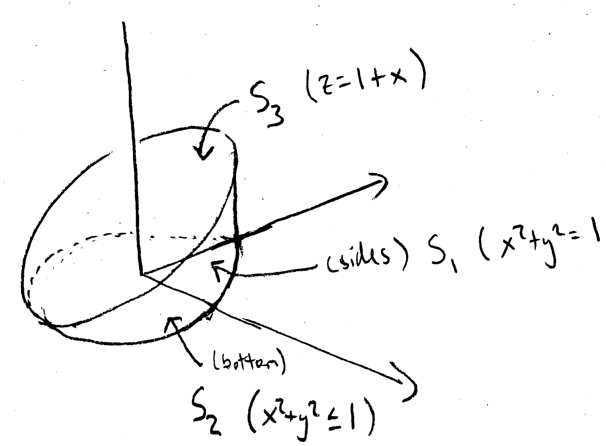


$\leftarrow S = S_1 + S_2 + S_3$

Then

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_{S_1} f(x,y,z)\,dS + \cdots + \iint_{S_n} f(x,y,z)\,dS}$$

EXAMPLE 3 Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2+y^2=1$, whose bottom is the disk $x^2+y^2\le 1$ in the plane $z=0$, and whose top S_3 is the part of the plane $z=1+x$ that lies above S_2.



To integrate $\iint_S z\,dS$, we integrate each side separately

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS.$$

Starting with S_1:

<u>Step 1</u> parametrize S_1, $x^2+y^2=1$

$$\left\{\begin{array}{ll} x=\cos\theta & \\ y=\sin\theta & \Big] \to 0 \le \theta \le 2\pi \\ z=z & \Big] \to z \text{ goes from } \underline{0} \text{ up to } \underline{1+x} \end{array}\right.$$

$$0 \le z \le 1+x = 1+\underline{\cos\theta}$$

So:

$$\left\{\begin{array}{l} \vec{r}(\theta,z) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + z\,\hat{k} \\ D = \{(\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le z \le 1+\cos\theta\} \end{array}\right.$$

<u>Step 2</u> Compute $|\vec{r}_\theta \times \vec{r}_z|$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + 0\hat{k}$$

$$|\vec{r}_\theta \times \vec{r}_z| = 1$$

<u>STEP 3</u> compute $\displaystyle\iint_{S_1} z\,dS = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,|\vec{r}_\theta \times \vec{r}_z|\,dz\,d\theta = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta$

$$= \int_0^{2\pi} \frac{(1+\cos\theta)^2}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(1 + 2\cos\theta + \underbrace{\cos^2\theta}_{\text{double angle!}}\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right) d\theta$$

$$= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi}$$

$$\boxed{= \frac{3\pi}{2}}$$

Next, we compute S_2. Notice that for S_2, $\underline{z=0}$, so

$$\iint_{S_2} z\,dS = \iint_{S_2} 0\,dS = 0$$

Next, S_3. Notice S_3 is the graph of the function $\overset{(=z)}{f(x,y)} = 1+x$, above $x^2+y^2 \le 1$. Thus, we know that $|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$.

$$\iint_{S_3} z\,dS = \iint_D \overset{z=1+x}{(1+x)} \cdot \sqrt{1 + 1^2 + 0^2}\, dA, \quad D = \{(x,y) : x^2+y^2 \le 1\}$$

use polar!

$$= \int_0^{2\pi}\int_0^1 (1 + r\cos\theta)\sqrt{2} \cdot r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (r + r^2\cos\theta)\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[\frac{r^2}{2} + \frac{r^3}{3}\cos\theta\right]_0^1 d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{3}\cos\theta\right) d\theta$$

$$= \sqrt{2} \cdot \left[\frac{1}{2}\theta + \frac{1}{3} \sin\theta \right]_0^{2\pi}$$

$$= \sqrt{2} \cdot \left[(\pi + 0) - (0 + 0) \right]$$

$$\boxed{= \sqrt{2}\, \pi}$$

Thus, we can conclude that

$$\iint_S z\, dS = \iint_{S_1} z\, dS + \iint_{S_2} z\, dS + \iint_{S_2} z\, dS = \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi$$

$$= \boxed{\frac{3\pi}{2} + \sqrt{2}\,\pi}$$

<u>Exercises</u> (16.6) #19, 23, 25, 33, 40, 49, 61

and if feeling bold: 64.

<u>Exercises</u> (16.7) #8, 9, 17