Lecture #16

(16.5)

Vector forms of Green's Theorem

REM Green's theorem

$$\left(\oint_C \vec{F} \cdot d\vec{r} \right) = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA ,$$

where C is the boundary of the plane region D, and $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$,

and $P(x,y)$, $Q(x,y)$ have continuous partials in an open region containing D.

REM $\operatorname{Curl}(\vec{F}(x,y,z)) = \nabla \times \vec{F}$, and if $\vec{F}(x,y,z) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath} + 0\hat{k}$,

then

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

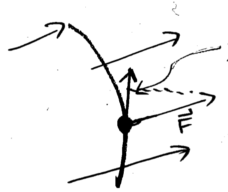
So Green's Theorem can be restated in vector form!

we have to dot with $\hat{k}$!

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint \operatorname{curl}(\vec{F}) \cdot \hat{k} \, dA}$$

EXERCISE Why dot $\operatorname{curl}(\vec{F})$ with $\hat{k}$? Hint: $\operatorname{curl}(\vec{F})$ is a vector, and you need a function in the integral.

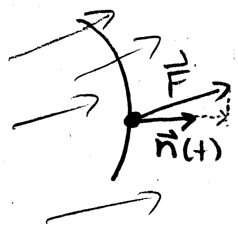
Next, remember that $\vec{F}\cdot d\vec{r}$ gives us the component of the vector field in the direction of the curve C:



$\frac{r'(t)}{|r'(t)|} = \vec{T}(t)$

(How much does $\vec{F}$ point along $r'(t)$?)

We can ask if there is anything meaningful to glean from looking at the component of $\vec{F}$ in the direction of the normal vector:



<u>REM</u> (from 126) If $\vec{r}(t) = \langle x(t), y(t) \rangle$, then

the unit tangent vector is $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{x'(t)}{|r'(t)|}\hat{\imath} + \frac{y'(t)}{|r'(t)|}\hat{\jmath}$

<u>EXERCISE</u> Show that

$$\vec{n}(t) = \frac{y'(t)}{|r'(t)|}\hat{\imath} - \frac{x'(t)}{|r'(t)|}\hat{\jmath} \quad \text{is normal to } \vec{T}(t).$$

<u>Hint</u>: use dot product...

<u>REM</u> $\vec{F}\cdot d\vec{r}$ is "short-hand" for $\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$,

which we first derived from $\vec{F}(\vec{r}(t))\cdot\vec{T}(t)\,ds$

$$\left(\vec{F}\cdot\vec{T}\,ds = \vec{F}\cdot\frac{r'(t)}{|r'(t)|}\cdot\underbrace{\sqrt{(x'(t))^2+(y'(t))^2}}_{\text{this is } |r'(t)| \text{ !!}}\,dt = \vec{F}\cdot\frac{r'(t)}{\cancel{|r'(t)|}}\cdot\cancel{|r'(t)|}\,dt \right)$$

i.e., " $\vec{F}\cdot d\vec{r}$ " came from $\vec{F}\cdot\vec{T}\,ds$, the amount of $\vec{F}$ in the direction of the unit tangent vector (i.e. the direction of the curve!)

So, we now want to consider

$$\underline{\vec{F}(t)\cdot\vec{n}(t)\,ds},$$

the amount of $\vec{F}$ in the <u>normal</u> direction! Let's just compute it.

Let $\vec{F} = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$, $\quad \vec{r}(t) = \langle x(t), y(t)\rangle$.

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C \vec{F}(\vec{r}(t))\cdot\left\langle \frac{y'(t)}{|r'(t)|}, \frac{-x'(t)}{|r'(t)|}\right\rangle \cdot \overbrace{\underbrace{\sqrt{(x'(t))^2+(y'(t))^2}}_{|r'(t)|\ !}\,dt}^{ds}$$

$$= \oint_C \langle P(x(t),y(t)),\ Q(x(t),y(t))\rangle \cdot \left\langle \frac{y'(t)}{|r'(t)|}, \frac{-x'(t)}{|r'(t)|}\right\rangle |r'(t)|\,dt$$

$$= \oint_C \left(\frac{P(x(t),y(t))\,y'(t)}{|r'(t)|} + \frac{-Q(x(t),y(t))\,x'(t)}{|r'(t)|}\right)|r'(t)|\,dt$$

$$= \oint_C P(x(t),y(t))\cdot\underbrace{y'(t)\,dt}_{dy} + -Q(x(t),y(t))\,\underbrace{x'(t)\,dt}_{dx}$$

$$= \oint_C P(x,y)\,dy - Q(x,y)\,dx$$

Apply (the usual) Green's Theorem!

(Careful, P and Q are backwards, and you have a negative!)

$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

$$= \iint_D \operatorname{div}(\vec{F})\,dA \qquad !!!$$

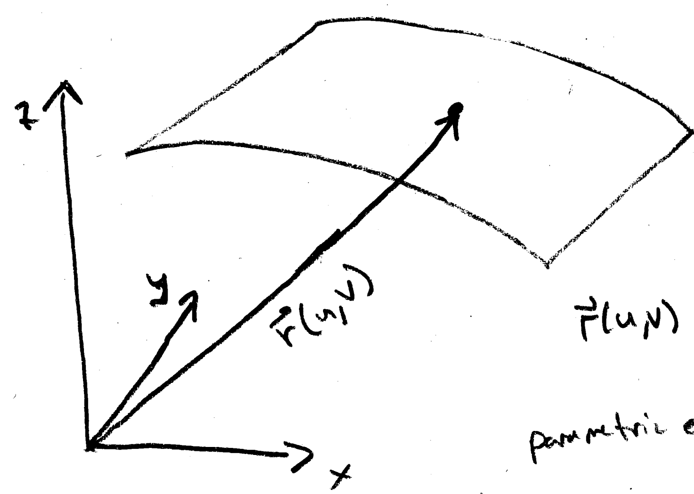
So, we get

$$\boxed{\oint_{C} \vec{F} \cdot \vec{n}\, ds = \iint_{D} \operatorname{div}(\vec{F})\, dA},$$

which is actually a 2-dimensional version of the divergence theorem!

16.6 Parametric Surfaces and Their Areas

Idea: We would like to parametrize a surface, similar to how we parametrize lines an curves in 3-D space. With lines and curves, we only need one parameter, but with surfaces, we need two. To remember this, you can think lines and curves are "1-dimensional" and surfaces are "two-dimensional".



$$\vec{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k}$$

parametric equations: $\begin{cases} x(u,v) \\ y(u,v) \\ z(u,v) \end{cases}$

EXAMPLE 1 Parametrize the surface $z = 4x^2 + 2y^2$.

Let

$$\begin{aligned} x &= u \\ y &= v \\ z &= 4u^2 + 2v^2 \end{aligned} \implies \vec{r}(u,v) = u\,\hat{\imath} + v\,\hat{\jmath} + (4u^2+2v^2)\,\hat{k}$$

EXAMPLE 2 Parametrize the top half of the cone $z = 5\sqrt{x^2+y^2}$

Same technique as example 1:

$$\begin{aligned} x &= u \\ y &= v \\ z &= 5\sqrt{u^2+v^2} \end{aligned} \implies \vec{r}(u,v) = u\,\hat{\imath} + v\,\hat{\jmath} + 5\sqrt{u^2+v^2}\,\hat{k}$$

Alternatively, notice we have an "x^2+y^2"

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= 5r \end{aligned} \implies \vec{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + 5r\,\hat{k}.$$

(+) $\begin{cases} 0 \le \theta \le 2\pi \\ r \ge 0 \end{cases}$

$$\begin{aligned} z &= 5\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} \\ &= 5\sqrt{r^2} \\ &= 5r \end{aligned}$$

EXAMPLE 3 What surface is described by $\vec{r}(u,v) = \cos(u)\,\hat{\imath} + \sin(u)\,\hat{\jmath} + v\,\hat{k}$?

$$\begin{aligned} x(u,v) &= \cos(u) \\ y(u,v) &= \sin(u) \\ z(u,v) &= v \end{aligned} \implies$$

NOTICE!

1. $x^2 + y^2 = 1$ always!

   $(\cos^2 u + \sin^2 u = 1)$

2. $z = v$, so we have no restrictions on z.

Thus, the surface being described is

$$x^2+y^2=1 \quad , \text{ a cylinder!}$$

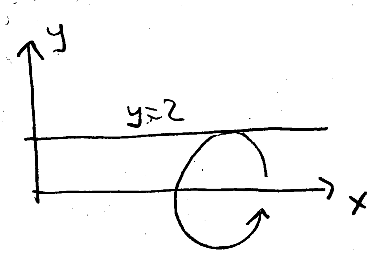
EXAMPLE 4 Parametrize $z^2+y^2=4$, $0 \le x \le 1$. (A piece of a cylinder.)

Let $x(u,v) = v$
$y(u,v) = 2\cos u$
$z(u,v) = 2\sin u$

$$\Rightarrow \left[\vec{r}(u,v) = v\,\hat{\imath} + 2\cos u\,\hat{\jmath} + 2\sin u\,\hat{k} \quad \text{for } \begin{cases} 0 \le u \le 2\pi \\ 0 \le v \le 1 \end{cases} \right]$$

REMARK Notice that the parametric equation can come with restrictions on u and v, such as in Example 4 where we only allow u to take on values between 0 and 2π, and v values between 0 and 1.

REMARK In the last couple examples, we could think of these cylinders as surfaces of revolution. For example, consider the cylinder $y^2+z^2=4$ from Ex 4. Start in the xy-plane with the function $y=2$:



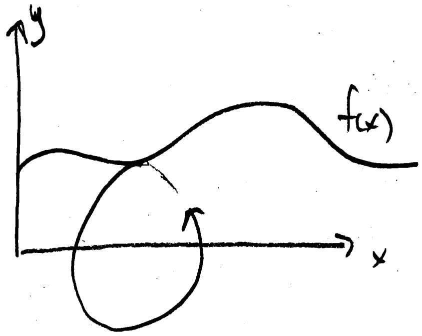
Rotate it about the x-axis, and we get a "Surface of Revolution." Now, think of $y=2$ as "$f(x)=2$", and notice:

$$\begin{cases} x = x \\ y = f(x)\cdot\cos(x) \\ z = f(x)\cdot\sin(x) \end{cases}$$

parametrizes the "surface of revolution."

Surface of Revolution

For a function $f(x)$ rotated about the x-axis, we can parametrize as follows:



$$\begin{cases} x = x \\ y = f(x)\cos\theta \\ z = f(x)\sin\theta \end{cases}$$

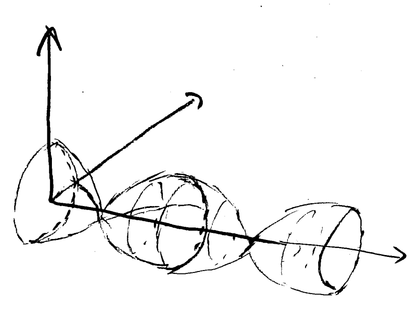
giving us $\left[\vec{r}(x,\theta) = x\,\hat{i} + f(x)\cos\theta\,\hat{j} + f(x)\sin\theta\,\hat{k} \quad \text{for } 0 \le \theta \le 2\pi \right]$.

EXAMPLE 5 Find parametric equations for the surface generated by rotating the curve $y = \cos(x)$, $0 \le x \le 2\pi$, about the x-axis. What does the surface look like?

$\Rightarrow f(x) = \cos(x)$

$$\begin{cases} x = x \\ y = \cos(x)\cdot\cos\theta \\ z = \cos(x)\cdot\sin\theta \end{cases} \quad \text{for } \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le x \le 2\pi \end{cases}$$

Looks like:



Back to general parametrizations!

EXAMPLE 6 Parametrize the sphere $x^2+y^2+z^2=a^2$.

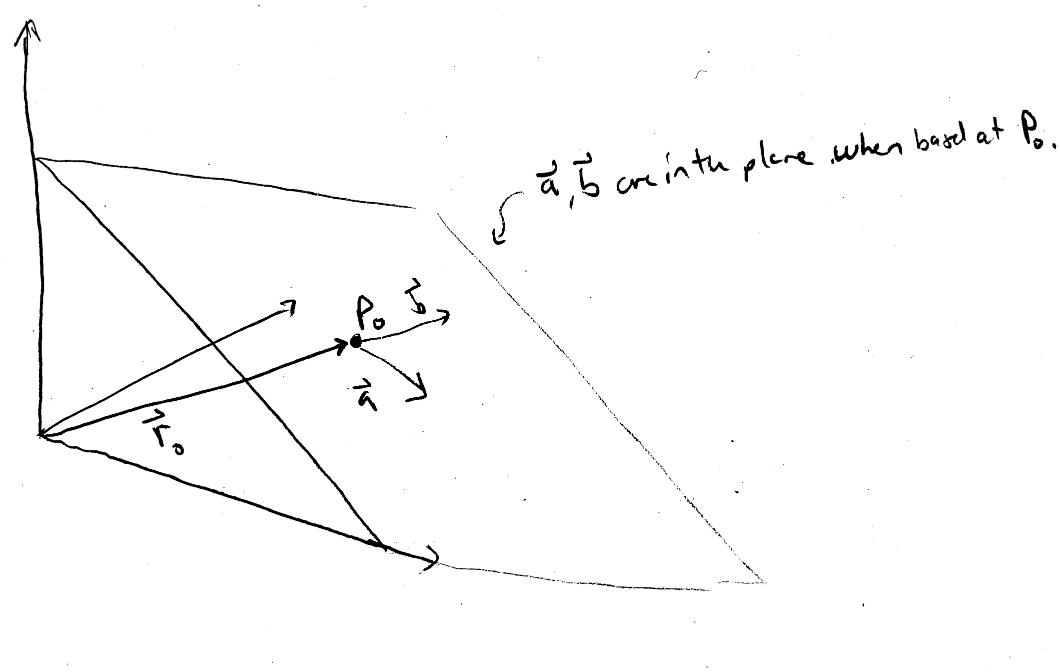
Use spherical coordinates! Notice, a is fixed.

$$x = a\sin\phi\cos\theta$$
$$y = a\sin\phi\sin\theta \qquad 0 \le \phi \le \pi$$
$$z = a\cos\phi \quad , \qquad 0 \le \theta \le 2\pi$$

$$\Rightarrow \quad \vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\hat{\imath} + a\sin\phi\sin\theta\,\hat{\jmath} + a\cos\phi\,\hat{k}$$

for $0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$

EXAMPLE 7 Find a vector function $\vec{r}(u,v)$ that represents a plane that passes through point $P_0=(x_0,y_0,z_0)$ with position vector $\vec{r}_0$ and that contains two non-parallel vectors $\vec{a}$ and $\vec{b}$, $\vec{a}=\langle a_1,a_2,a_3\rangle$ and $\vec{b}=\langle b_1,b_2,b_3\rangle$.



Then, notice that if I start at $P_0 = (x_0, y_0, z_0)$, I can move along the "$\vec{a}$" direction or "$\vec{b}$" direction, or a little of both, to get anywhere on the plane! So

$$\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}.$$

Where u is how much you move in the $\vec{a}$ direction, and v is how much you move in the $\vec{b}$ direction.

EXERCISE If $\vec{r}_0 = \langle \cancel{x}_0, y_0, z_0 \rangle$, $\vec{a} = \langle a_1, a_2, a_3 \rangle$, and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, what is $\vec{r}(u,v)$ in the form $\langle x(u,v), y(u,v), z(u,v) \rangle$?

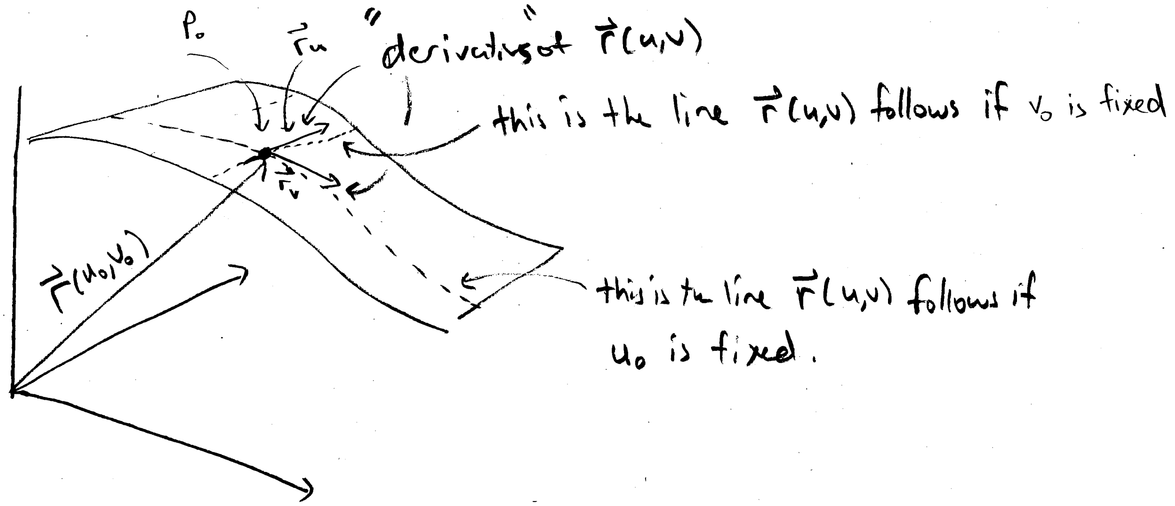
In other words, what are $x(u,v)$, $y(u,v)$ and $z(u,v)$?

Hint: $x(u,v) = x_0 + ua_1 + vb_1$.

Tangent Planes

Given any parametric surface $\vec{r}(u,v)$ and a point on that surface, can we write the equation of the tangent plane at that point?

Yes!



rem $r'(t)$, on a curve, is a vector tangent to the curve!

So, it is reasonable to expect

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\,\hat{\imath} + \frac{\partial y}{\partial u}(u_0,v_0)\,\hat{\jmath} + \frac{\partial z}{\partial u}(u_0,v_0)\,\hat{k}$$

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\,\hat{\imath} + \frac{\partial y}{\partial v}(u_0,v_0)\,\hat{\jmath} + \frac{\partial z}{\partial v}(u_0,v_0)\,\hat{k}$$

are both tangent to the surface (in two different directions)!

In other words, these two vectors live in the tangent plane! (at the point (x_0, y_0, z_0) given by $\vec{r}(u_0,v_0)$.)

$\left(\text{so} \Rightarrow \vec{r}(\tilde{u},\tilde{v}) = \vec{r}(u_0,v_0) + \tilde{u}\,\vec{r}_u + \tilde{v}\,\vec{r}_v\right)$

EXAMPLE 8 Find the tangent plane to the surface with parametric equations $x=u^2$, $y=v^2$, and $z=u+2v$ at the point $(1,1,3)$.

$$\vec{r}(u,v) = u^2\,\hat{i} + v^2\hat{j} + (u+2v)\,\hat{k}$$

$$\vec{r}_u = 2u\,\hat{i} + 0\hat{j} + (1)\hat{k}$$

$$\vec{r}_v = 0\,\hat{i} + 2v\hat{j} + 2\hat{k}$$

These vectors will live in the tangent plane for any fixed u,v, so lets get a normal vector!

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,\hat{i} + 4u\hat{j} + 4uv\,\hat{k}$$

So, our normal vector is $-2v\,\hat{i} - 4u\hat{j} + 4uv\,\hat{k}$.

We need the point on the surface: given $(1,1,3) = (x_0, y_0, z_0)$.

What is the corresponding (u,v)?

$$x = u^2 \Rightarrow u = \pm 1$$

$$y = v^2 \Rightarrow v = \pm 1$$

$$z = u + 2v \Rightarrow u = 1,\ v = 1$$

So $\vec{r}(1,1)$ points to $(1,1,3)$.

$\Rightarrow$ normal vector: $-2v\,\hat{i} - 4u\hat{j} + 4uv\,\hat{k} \xrightarrow[v=1]{u=1} -2\hat{i} - 4\hat{j} + 4\hat{k}$.

point: $(1,1,3)$

normal vector: $\langle -2, -4, 4 \rangle$

$\longrightarrow$ $\boxed{\text{equation of plane}}$ (126)

$$-2(x-1) + (-4)(y-1) + 4(z-3) = 0$$

$$\boxed{x + 2y - 2z = -3}$$

* see equation of a plane handout if needed!