16.5 Curl and Divergence

DEF curl, "takes in a vector field, spits out a vector field."

$$\text{curl } F := \nabla \times \vec{F}, \quad \text{where } \vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k},$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

REMARK This equation works in 2 or 3 dimensions. In 2 dimensions,

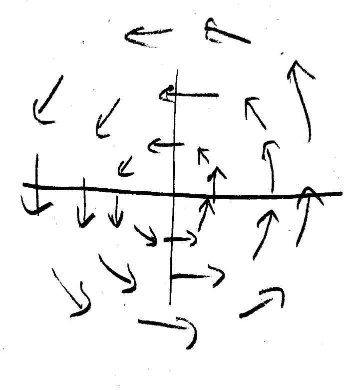
$F(x,y) = P\hat{\imath} + Q\hat{\jmath}$, so $\text{curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$.

EXAMPLE 1 Find curl $\vec{F}$ where $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + (1-{-1})\hat{k}$$

$$\Rightarrow \text{curl } \vec{F} = 2\hat{k}$$

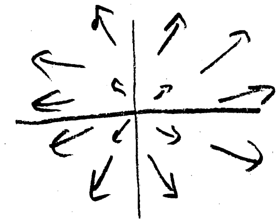
pic:



EXAMPLE 2 Find $\text{curl}\,\vec{F}$ where $\vec{F} = 2x\,\hat{\imath} + 2y\,\hat{\jmath}$.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 0 \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

pic:



$\text{curl}\,\vec{F} = \vec{0}$.

Notice! $\vec{F} = \nabla f$ where $f = x^2 + y^2$.

REMARK If we look at the two vector fields from the last two examples, we see that the first example is not conservative, and the vector field has some sort of "twist" around O. In the second example, there is no "twist", and the vector field is conservative.

From this, we might guess that if the curl of a vector field is non-zero, that there is some sort of "twist" in the vector field. If a vector field has this "twist", can it be conservative? No! (Think this through!)

THM "If $\vec{F}$ conservative, then $\text{curl}\,\vec{F} = \vec{0}$" more precisely, If f is a function of 2 or 3 variables that has 2^{nd}-order partial derivatives, then

$$\boxed{\text{curl}(\nabla f) = \vec{0}}$$

REMARK This theorem helps us identify when a vector field is <u>not</u> conservative.

$$\text{If conservative} \Rightarrow \text{curl}\,\vec{F} = \vec{0}$$

means

$$\text{If curl}\,\vec{F} \neq \vec{0} \Rightarrow \vec{F} \text{ is } \underline{\text{not}} \text{ conservative.}$$

Exercise Show that the theorem above, when in 2-dimensions, is just a "fancy" restatement of the theorem

$$\text{"If conservative} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\text{"}$$

(<u>Hint</u>: Compute $\text{Curl}(\nabla f)$ for $\nabla f = \underset{\frac{\partial f}{\partial x}}{\underset{\shortparallel}{P}}\,\hat{\imath} + \underset{\frac{\partial f}{\partial y}}{\underset{\shortparallel}{Q}}\,\hat{\jmath}$)

In other words, this theorem brings us into the third dimension!

THM If $\vec{F}$ is a vector field defined on an open (!) simply connected region V and $\text{Curl}\,\vec{F} = \vec{0}$, then $\vec{F}$ is conservative.
(and whose components have continuous partial derivatives on the region!)

REMARK You do not need to worry about what simply connected means in 3-dimensions. However, you should know $\mathbb{R}^3$ is simply connected, so

COR If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ and $\vec{F}$ has components whose partials are continuous on $\mathbb{R}^3$, and $\text{curl}\,\vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

EXAMPLE 3 a) Show that $\vec{F}(x,y,z) = y^2z^3\hat{\imath} + 2xyz^3\hat{\jmath} + 3xy^2z^2\hat{k}$ is a conservative vector field.

1. $\vec{F}$ is defined on all of $\mathbb{R}^3$

2. All partial are continuous! We will see this (for b) when computing curl $(\vec{F})$.

the other 3 are computed here!

$\overset{0}{\overset{\shortparallel}{\frac{\partial P}{\partial x}}}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}$ ✓

$\frac{\partial R}{\partial y} = 2xz^3$ ✓, $\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z}$ ✓

$\frac{\partial R}{\partial z} = 6xy^2z$, $\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}$ ✓

3. $$\text{curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\hat{\imath} + (3y^2z^2 - 3y^2z^2)\hat{\jmath} + (2yz^3 - 2yz^3)\hat{k}$$

$$= \vec{0}$$

$\Rightarrow$ $\vec{F}$ is conservative.

b) Find a function f such that $\vec{F} = \nabla f$.

① $f_x(x,y,z) = y^2z^3$

② $f_y(x,y,z) = 2xyz^3$

③ $f_z(x,y,z) = 3xy^2z^2$

$\Rightarrow$ Integrate ① w.r.t. x: $f(x,y,z) = \int_0^x (y^2z^3)\,dx = \overbrace{xy^2z^3 + \underline{g(y,z)}}^{\text{Proposed } f(x,y,z)}$

(Constant!)

Proposed $f(x,y,z) = xy^2z^3 + g(y,z)$

Now, differentiate the proposed function w.r.t. y :

$$f(x,y,z) = xy^2z^3 + g(y,z)$$

$$f_y(x,y,z) = \underbrace{2xyz^3}_{\substack{\| \\ f_y(x,y,z) \\ (\text{see } ②)}} + g_y(y,z)$$

$$\Rightarrow g_y(y,z) = 0$$

$$\Rightarrow g(y,z) = h(z)$$

only a function of z !

Now : $f(x,y,z) = xy^2z^3 + h(z)$

$$f_z(x,y,z) = 3xy^2z^2 + h'(z)$$

Compare with ③ : Need $3xy^2z^2 = 3xy^2z^2 + h'(z)$

$$\Rightarrow h'(z) = 0$$

$$\Rightarrow h(z) = C, \text{ constant !}$$

Thus, $\boxed{f(x,y,z) = xy^2z^3 + C}$, for any C we want.

REMARK The curl takes in a vector field and spits out a vector field. The vector field may or may not be the 0 vector field ($\vec{0}$ at each point). If not, the vector field may be $\vec{0}$ at certain points, and non-zero at others. What does this mean?

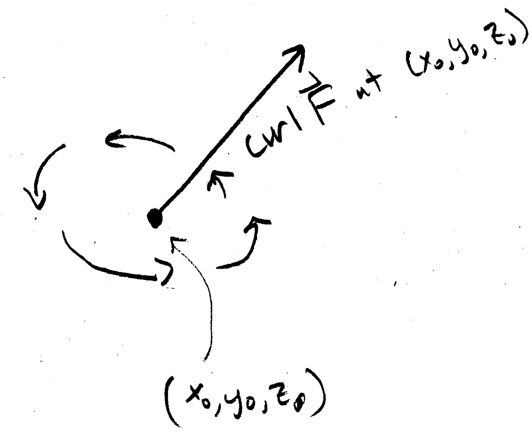
If $\text{curl}(\vec{F})(x_0,y_0,z_0) = \vec{0}$ at the point (x_0,y_0,z_0),

evaluated at a point!

we say that the vector field is <u>irrotational</u> at (x_0,y_0,z_0).

In other words, no twist at the point!

<u>Question</u> Where is the curl vector field pointing?
In the direction orthogonal to the rotation at a point! The bigger the vector, the faster the particles move around the point!



e.g.
compare curl of
① $\vec{F} = \langle -y, x\rangle$
② $\vec{G} = \langle -100y, 100x\rangle$.

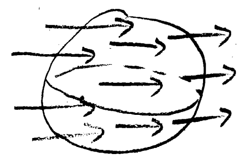
<u>DEF</u> Divergence of $\vec{F}$

If $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, and $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ exist, then

$$\text{div}\,\vec{F} := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R\rangle = \text{“}\nabla \cdot \vec{F}\text{”}$$

<u>REMARK</u> What is the divergence? It is a way of measuring how much "fluid" diverges from a point. It is kind of hard to imagine at a point, so think of a small ball in space:



Divergence is telling you about the "net rate of change" through this small ball. This is sort of like a "flux." If $\operatorname{div} \vec{F} = 0$, then we say $\vec{F}$ is <u>incompressible</u>. (You can't compress the fluid at points!)

<u>EXERCISE</u> Compute $\operatorname{div}(\operatorname{curl} \vec{F})$.

<u>THM</u> If $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, R have continuous <u>second-order</u> partial derivatives, then

$$\operatorname{div}(\operatorname{curl} \vec{F}) = \underline{\qquad}$$

← Fill in after exercise!

<u>REMARK</u> This tells us that if $\operatorname{div}(\vec{G}) \neq \underline{\quad}$, then $\vec{G}$ is not the curl of another vector field!

Exercise EXAMPLE 5 in the text (16.5).

DEF Laplacian

"divergence of the gradient"

$$\operatorname{div}(\nabla f) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

We call $\nabla \cdot \nabla = \nabla^2 = \Delta$ the *Laplacian operator*.

EXERCISES Try the following book problems:

#9-11, #12, #31, #38.

Math / Physics Majors may enjoy

#33-36

EXERCISE (For the bold)

Show that any continuous function f on $\mathbb{R}^3$ is the divergence of some vector field $\vec{G}$.

[Hint: Let $\vec{G} = \langle g(x,y,z), 0, 0 \rangle$, where

$$g(x,y,z) = \int_0^x f(t,y,z)\,dt.$$]