Lecture #14

16.4 Green's Theorem

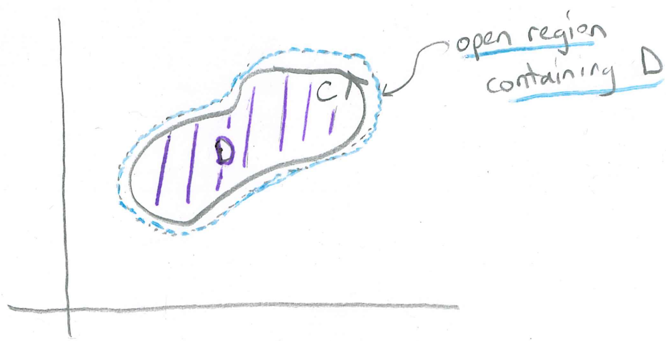
DEF (positive orientation)

We say a positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C.

THM (Green's Theorem)

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



REMARKS

① Green's Theorem is a statement about vector fields!
($\vec{F} = \langle P, Q \rangle$).

② It may not look like it, but this is yet another generalization of the Fundamental Theorem of Calculus.

REM

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

Notice that the left-hand side is evaluation along boundary points (which in this case are just a and b). The right side is an integral of a derivative of that function. Compare that with Green's Theorem.

③ We will see two other generalizations of the Fundamental Theorem of Calculus, and in each one, we will see the same relationship between "boundary" and "derivatives". This is suggestive of a bigger picture just out of sight...

(EXAMPLES START ON
THE NEXT PAGE!)

NOTATION

$\oint_C$ or $\oint_C$ mean C is a closed curve with positive orientation.
(changes meaning by author - careful!)

EXAMPLE 1

Evaluate $\oint_C (3y - e^{\sin(x)})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2+y^2=9$.

NOTE: $\oint_C$ means C is positively oriented, which is good for Green's Theorem!

STEP 1: Do NOT parametrize! Instead, make sure $P(x,y)$ and $Q(x,y)$ are well-defined on the circle and the inside of the circle (KEY!).

$$P(x,y) = 3y - e^{\sin(x)}$$

$$Q(x,y) = 7x + \sqrt{y^4+1}$$

You need for partial derivatives to exist on the entire disk. (They do!)

STEP 2: Apply Green's Theorem

$$\frac{\partial P}{\partial y} = 3, \quad \frac{\partial Q}{\partial x} = 7, \quad D = \{(x,y) : x^2+y^2 \le 9\}$$

$$\oint_C (3y - e^{\sin(x)})\,dx + (7x + \sqrt{y^4+1})\,dy = \iint_D (7-3)\,dA$$

$$= 4 \cdot \iint_D dA$$

$$= \boxed{36\pi}$$

EXAMPLE 2 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

SOLUTION: Could compute $\iint_D dA$, where D is the area enclosed by the ellipse, but we could also do this using Green's Theorem. Notice, if we let

$$P(x,y) = \frac{-1}{2} y$$

$$Q(x,y) = \frac{1}{2} x$$

Then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} - \frac{-1}{2} = 1$.

(EXERCISE What other functions P, Q have this property?)

Now, notice, we can use Green's Theorem!

$$A(D) = \iint_D 1 \cdot dA = \oint_C -\frac{1}{2} y \, dx + \frac{1}{2} x \, dy$$

(where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.)

Then $A(D) = \frac{1}{2} \oint_C x\,dy - y\,dx$

parametrize:
$x(t) = a\cos t$ (*)
$y(t) = b\sin t$

$$= \frac{1}{2} \int_0^{2\pi} \underbrace{(a\cos t)}_{x(t)} \cdot \underbrace{(b\cos t)}_{y'(t)} dt - \underbrace{(b\sin t)}_{y(t)} \cdot \underbrace{(-b\cos t)}_{x'(t)} dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab(\cos^2 t + \sin^2 t)\, dt$$

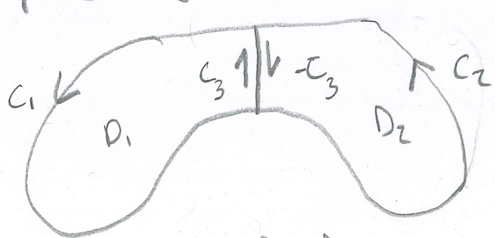
$$= \frac{1}{2} ab \int_0^{2\pi} dt = \boxed{\pi ab}$$

REMARK The last example shows how we can use Green's Theorem to compute an area using a line integral.

EXTENDING GREEN'S THEOREM

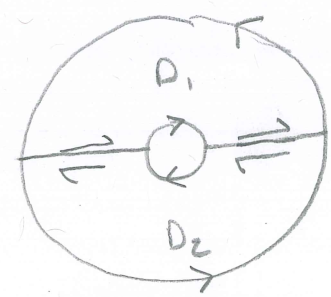
We can apply Green's Theorem to any set of finite simple regions. (By simple region, we just mean a region that can be integrated over with respect to x or y first and only yield one integral, i.e. it doesn't split into two integrals!)

How? Break the region into pieces, each of which is a simple region:



$D = D_1 + D_2$

or



$D = D_1 + D_2$

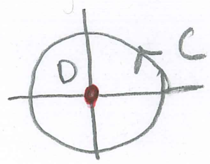
In fact, we can employ this idea in order to use Green's Theorem on a line integral that includes a point on its interior where the vector field is undefined.

For example, we might want to integrate

$$\oint_C \frac{-Ky}{(x^2+y^2)^{3/2}}\,dx + \frac{Kx}{(x^2+y^2)^{3/2}}\,dy \quad \text{over } C: x^2+y^2=1.$$

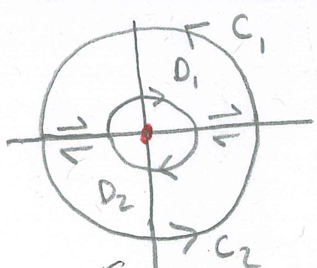
But $\vec{F} = \left\langle \frac{-Ky}{(x^2+y^2)^{3/2}}, \frac{Kx}{(x^2+y^2)^{3/2}} \right\rangle$ is undefined at $(0,0)$, and so are its partial derivatives! (Check this ...)

So, we can't use Green's theorem:



, D region enclosed by C.

But, we can get at this another way. We can use Green's theorem on the top and bottom half... with one adjustment.

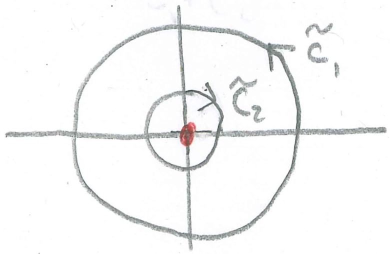


(using Green's Theorem)

$$\oint_{C_1} = \iint_{D_1}, \qquad \oint_{C_2} = \iint_{D_2}$$

(Assume C_2 is the circle $x^2+y^2=\frac{1}{4}$, radius $\frac{1}{2}$)

reduces to (notice the cancellation!)

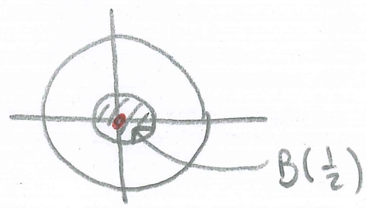


$$\int_{C_1+C_2} = \iint_{D_1+D_2}$$

and

$$\int_{C_1+C_2} = \int_{\tilde{C}_1+\tilde{C}_2}, \quad \text{so}$$

$$\int_{\tilde{C}_1+\tilde{C}_2} = \iint_{D_1+D_2} = \iint_{D} - \iint_{B(\frac{1}{2})}.$$



Then, notice $\tilde{C}_1 = C$, so

$$\oint_{\tilde{C}_1} + \int_{C_2} = \iint_{D} - \iint_{B(\frac{1}{2})}$$

$$\oint_{C} = \iint_{D} - \iint_{B(\frac{1}{2})} - \int_{C_2}.$$

If something simplifies, this could help us say something ...

<u>EXERCISE</u> Work EXAMPLE 5 in 16.4.