Lecture #13

16.3 Fundamental Theorem for Line Integrals

REM Fundamental Theorem of Calculus, part 2

$$\int_a^b F'(x)\,dx = F(b) - F(a),$$

where F' is continuous on $[a,b] = \{x \text{ such that } a \le x \le b\}$.

EXERCISE Write the integral above as a line integral through a 1-dimensional vector field. Hint: Let $F'(x)$ be the component function for a vector field (with 1-dim).

THM Fundamental Theorem for Line Integrals

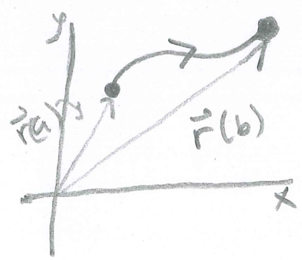
Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

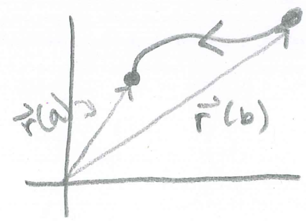
<u>REMARK</u> We should notice a few things:

① This says that if the vector field we are integrating over is the gradient of a surface (i.e. is a <u>conservative vector field</u>!), then the <u>path between the endpoints</u> of the curve <u>does not matter</u>! Only the endpoints matter.

② Second, orientation or direction of the parametrization <u>matters</u>. It matters whether we start at $\vec{r}(a)$ and go to $\vec{r}(b)$ (as in the statement of the theorem), or start at $\vec{r}(b)$ and go to $\vec{r}(a)$.



vs.



③ Lastly (as hinted at in ①), recall that <u>conservative vector fields</u> are vector fields that are the gradient of a surface. So, we can read the theorem as a statement about <u>conservative vector fields</u>. The theorem says that if $\vec{F} = \nabla f$ for some f, then the line integral will be the net change in f from $\vec{r}(a)$ to $\vec{r}(b)$.

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proof of THM :

Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$; for $a \le t \le b$.

Then compute:

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt \quad , \text{ by definition}$$

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \Big|_{\vec{r}(t)} \cdot \langle x'(t), y'(t), z'(t) \rangle\,dt$$

plug in $x = x(t)$, $y = y(t)$, $z = z(t)$ into the gradient.

$$= \int_a^b \left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} \right) dt$$

chain rule backwards!

$$= \int_a^b \frac{d}{dt} \Big(f(x(t), y(t), z(t)) \Big)\,dt$$

"$f(\vec{r}(t))$"

$$\overset{t=b}{=} f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)).$$

$\blacksquare$

eg: If: $f(x,y,z) = xyz^2$, $x(t) = t$, $y(t) = 3t$, $z(t) = \sqrt{t}$

then: $f(x(t), y(t), z(t)) = 3t^3$

$f(x(t), y(t), z(t))$ is just a function in t! We can use the Fundamental Theorem of Calculus!

<u>EXERCISE</u> Understand each line in the proof. Be able to reproduce it.

<u>EXAMPLE 1</u> Find the work done by the vector field

$$\vec{F}(x,y) = \langle 2xe^y, e^y x^2 - \sin(y) \rangle$$

on a particle when moving it from $(-2,2)$ to $(3,17)$ along the parabola $y = x^2 + 2x + 2$.

We could compute $\int_C \vec{F} \cdot d\vec{r}$ directly, using a parametrization and our formula. <u>BUT</u> there is a <u>MUCH</u> <u>EASIER</u> way. We should first check to see if $\vec{F}$ is a conservative vector field.

Assume $\vec{F} = \nabla f$, and try to find f.

Then:

$$\vec{F} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

so

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xe^y \\[2ex] \dfrac{\partial f}{\partial y} = e^y x^2 - \sin(y). \end{cases}$$

Now, integrate $\frac{\partial f}{\partial x}$ with respect to x:

$$\int \frac{\partial f}{\partial x}\,dx = \int 2xe^y\,dx = x^2e^y \left(+\underline{\underline{g(y)}}\right)$$

Then,

$$f(x,y) = x^2e^y + \underline{\underline{g(y)}}$$

there could be a function of y that $\frac{\partial f}{\partial x}$ doesn't see!

(If you take the partial derivative with respect to x for our proposed $f(x,y)$ above, $g(y)$ vanishes!)

Now, compute $\frac{\partial f}{\partial y}$ for the proposed $f(x,y)$:

$$\frac{\partial f}{\partial y} = x^2e^y + g'(y)$$

And notice $\frac{\partial f}{\partial y} = e^yx^2 - \sin(y)$ by our assumption.

Then, we see that $\boxed{g'(y) = -\sin(y)}$.

Integrating, we see $g(y) = \cos(y) + C_1$, so

$$\boxed{f(x,y) = x^2e^y + \cos(y) + C_1}$$

for $\underline{\underline{\text{any}}}$ C_1 is a valid function!

Next, we can use $f(x,y)$ to evaluate the integral using the fundamental theorem of line integrals!

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = \overset{\text{endpoint}}{f(3,17)} - \overset{\text{starting point}}{f(-2,2)}$$

$$= \left(3^2 \cdot e^{17} + \cos(17) + C_1\right)$$
$$- \left(4e^2 + \cos(2) + C_1\right)$$

$$= \boxed{9e^{17} + \cos(17) - 4e^2 - \cos(2).}$$

<u>Notice</u> (1) The choice of C_1 didn't matter!

(2) We didn't have to parametrize the curve C!

<u>RECALL</u> Examples 3 and 4 from the lecture notes on 16.2. In general, for curves C_1 and C_2 with the same end points, but different paths

$$\int_{C_1} \vec{F}\cdot d\vec{r} \neq \int_{C_2} \vec{F}\cdot d\vec{r},$$

BUT the Fundamental theorem tells us that for conservative vector fields,

$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$$

where C_1 and C_2 are as described on the previous page.

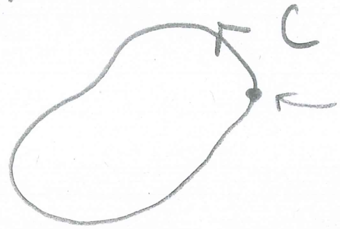
DEF If $\vec{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path if

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$ for any two paths C_1 and C_2 in D that have the same initial points and the same terminal points.

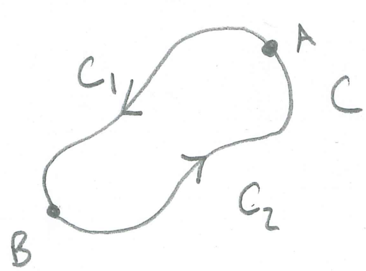
NOTE If $\vec{F}$ is conservative, the Fundamental Theorem of Line Integrals tell us $\int_C \vec{F} \cdot d\vec{r}$ is independent of path. Is the converse true? If a line integral is independent of path, is the corresponding vector field conservative?

We will try to answer this. To do this, we will give another characterization of "independent of path." We will need a few definitions to make this work.

DEF A curve C is called <u>closed</u> if its terminal point is the same as its initial point, i.e. $\vec{r}(a) = \vec{r}(b)$.



NOTICE: We can pick two points on a closed curve C, and think of C as the composition of two curves C_1 and C_2:



- C_1 goes from A to B
- C_2 goes from B to A

"$C = C_1 + C_2$"

Then, if we have a line integral that is <u>independent of path</u>, we can compute the line integral over C.

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r}$$

REM: Changing the orientation/direction we move along a curve gives us a negative!

Vector field helps!

vs.

vector field pushes against us!

C_2 goes from B to A, so $-C_2$ goes from A to B, same as C_1! So these integrals are the same!

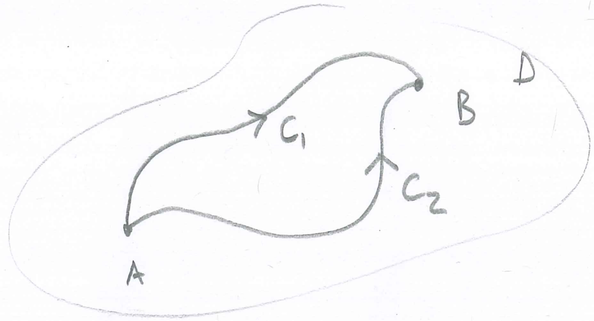
$$= \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_1} \vec{F}\cdot d\vec{r}$$

$$= 0$$

So, we just showed that if line integrals in a domain D are <u>independent of path</u>, then $\int_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve in D! Is the converse true?

If $\int_C \vec{F}\cdot d\vec{r} = 0$ for any closed path in D, are line integrals in D independent of path?

Pick any two points A and B in D, and pick any two paths from A to B:

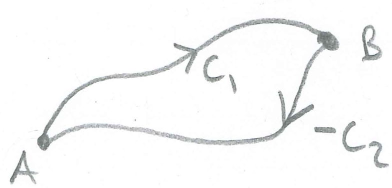


(D is a subset of $\mathbb{R}^2$)

If we assume $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed loop C in D,

can we show that $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$? (Notice, since we picked A and B at random, and the curves C_1 and C_2, arbitrarily, if we can show what is asked above, we will have shown that if $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve in D, then line integrals are independent of path.)

Notice $C_1 + -C_2$ is a closed loop!



So, we know:

$$0 = \int_{C_1 + -C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}.$$

Thus, we can conclude

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}.$$

Nice! Let's capture what we have done in a theorem.

<u>THM</u> $\int_{\tilde{C}} \vec{F}\cdot d\vec{r}$ is independent of path if and only if $\int_C \vec{F}\cdot dr = 0$ for every closed path C (in D, where $\vec{F}$ is cont. in D)

Why did we do that? We can use this theorem to show:

<u>THM</u> Suppose $\vec{F}$ is a vector field that is continuous on an <u>open</u>, <u>connected</u> region D. If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \vec{F}$.

defined below

In other words, we now have three equivalent characterizations:

$$\vec{F} \text{ is a conservative vector field on } D, \text{(open, connected)}$$

$$\Updownarrow$$

$$\int_C \vec{F} \cdot d\vec{r} \text{ is independent of path}$$

$$\Updownarrow$$

$$\int_C \vec{F} \cdot d\vec{r} = 0 \text{ for any closed curve } C.$$

Now, let's clarify this issue with the domain D... it needs to be open and connected.

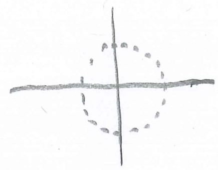
<u>DEF</u> By <u>open</u>, we mean that for each point p in the domain, there is a small disk with center p that lies completely in D:

i.e. D does not have a "boundary"

If D contained a point on the "boundary", no disk containing p at the center is in D!

EXAMPLES · open interval (a,b) is open. (Does not contain a or b).

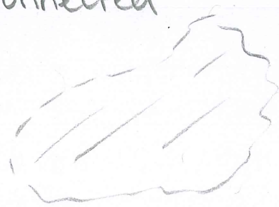
· disk: $D = \{x : x^2+y^2 < 1\}$ ← does not contain points on $x^2+y^2=1$.



DEF By connected, we mean that any two points in D can be joined by a path in D.

(later, this is actually the definition of path-connected, not connected... but that's a different class.)

Connected



not connected

Proof of THM: See text! (starts on p. 1129)

(I will try to add a sketch of this proof at the end of these notes.)

So this answers the question on the bottom of page 7 of these notes, but it leads to many more. We now have a few characterizations of what it means to be conservative, but it would be nice if we could find another condition that is easy to check.

<u>REM</u> If $f(x,y)$ is smooth (or at least twice differentiable, with continuous 2^{nd} derivatives), then

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}}$$

("Mixed Partials" are "equal".)

<u>REMARK</u> If a vector field is conservative, then if we let

$$\vec{F} = \langle P(x,y), Q(x,y) \rangle$$

we know

$$\vec{F} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

for some f, so

$$P = \frac{\partial f}{\partial x}$$

$$Q = \frac{\partial f}{\partial y}.$$

Then, take a derivative of P with respect to y and one of Q with respect to x:

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

and we see that

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

Let's capture this in a theorem.

<u>THM</u> If $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ is a conservative vector field, where P and Q have continuous first order partial derivatives on a domain D, then throughout D, we have

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}.$$

(If $\vec{F}$ conservative, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.)

So this is a property of conservative vector fields, but it doesn't tell us how to identify a conservative vector field. <u>If</u> the converse of the theorem were true, then it would! (If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then conservative is the converse.) <u>BUT</u> this is <u>not</u> true!

We need something else ...

THM Let $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ be a vector field on an open, $\boxed{\text{simply-connected}}$ region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D. Then $\vec{F}$ is conservative.

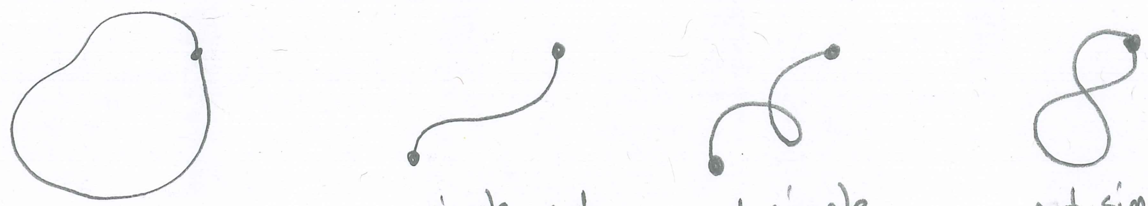
(If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and D is open and simply-connected, then $\vec{F}$ is conservative.)

REMARK The key here is that D is simply connected, which we will define below.

REMARK We won't prove this theorem yet. Once we have Green's theorem, we will see it as a simple consequence.

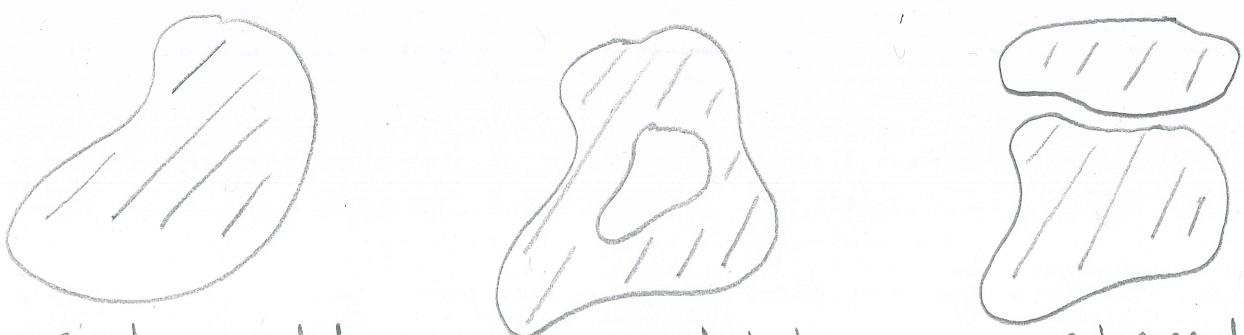
Now, lets say what we mean by simply-connected. We will need a few definitions.

DEF We say a simple curve is a curve that does not intersect itself anywhere between endpoints.



simple, closed — simple, not closed — not simple, not closed — not simple, closed

DEF A simply-connected region is a connected region such that every simple closed curve in D encloses only points that are in D.



simply-connected region — connected, but not simply connected — not connected, so not simply-connected.

EXAMPLE 2 Is $F(x,y) = (x-y)\,\hat{\imath} + (x-2)\,\hat{\jmath}$ conservative?

If $\vec{F}$ is conservative, then we would know that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Then, we can say that if $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, there is no way $\vec{F}$ is conservative!

Notice, $\begin{cases} P(x,y) = x-y \\ Q(x,y) = x-2 \end{cases}$

and $\begin{cases} \frac{\partial P}{\partial y} = -1 \\ \frac{\partial Q}{\partial x} = 1 \end{cases}$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $\vec{F}$ is not conservative!

(If we saw that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ... does it mean $\vec{F}$ is conservative? We need to check the domain first! Is it simply-connected?)

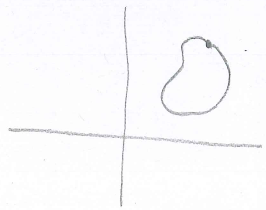
EXAMPLE 3 Is $\vec{F}(x,y) = (3+2xy)\,\hat{i} + (x^2-3y^2)\,\hat{j}$ conservative?

Notice, $P(x,y) = 3+2xy$ and $Q(x,y) = x^2-3y^2$

and ...

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}.$$

This does not mean $\vec{F}$ is conservative! We must first check that $\vec{F}$ is defined on a simply connected region. Since the domain of $\vec{F}$ is all of $\mathbb{R}^2$, we are really asking is if $\mathbb{R}^2$ is simply-connected.



It is! (Check the definition!). $\mathbb{R}^2$ is also open (check the definition). Now, we can conclude that $\vec{F}$ is conservative.

EXERCISE Read and work EXAMPLES 4 and 5 in the text. These examples show the strengths + limitations of the theorems we have.

EXERCISE ① Is $\vec{F}(x,y) = -y\,\hat{\imath} + x\,\hat{\jmath}$ a conservative vector field?

② Is $\vec{F}(x,y) = \dfrac{x}{(x^2+y^2)^{3/2}}\,\hat{\imath} + \dfrac{y}{(x^2+y^2)^{3/2}}\,\hat{\jmath}$ a conservative vector field?

EXERCISE (True/False)

1. Let C be any closed path on a conservative vector field. Then $\int_C \vec{F} \cdot d\vec{r} = 0$.

2. Line integrals on vector fields depend only on the initial point and terminal point of a curve.

3. A line integral is independent of path on a domain D if and only if the vector field on D is conservative.

4. Let $\vec{F}$ be a vector field such that it is smooth on all of $\mathbb{R}^2$. If $\vec{F}(x,y) = M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath}$ and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $\vec{F}$ is conservative.

5. Let $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$ on an open, connected domain D. If $\vec{F}$ is conservative, then $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$.

6. If line integrals over a vector field $\vec{F}$ are

independent of path on a domain D, and $\vec{F} = M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath}$, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

7. Suppose $\int_C \vec{F}\cdot d\vec{r} = 0$ on every closed loop in the domain $D = \{(x,y) : \frac{1}{2} < x^2+y^2 < 1\}$. Then $\vec{F}$ is a conservative vector field on D.

8. Suppose $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on the open, connected region D where $\vec{F}$ is defined and smooth. Then $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$, where C_1 and C_2 are curves in D with the same endpoints.

9. Suppose $\vec{F}(x,y) = M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath}$ where $\vec{F}$ is smooth on a simply connected region D and $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$. Then on any closed loop C in D, $\int_C \vec{F}\cdot d\vec{r} = 0$.

10. Suppose $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$ is a differentiable vector field on an open, connected region D and $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$. Then $\vec{F}$ is not conservative on D, but $\vec{F}$ could be on a slightly larger region $\tilde{D}$ that contains D.

"Cheat Sheet" on the next page!

"Cheat Sheet"

We have the following implications:

① Fundamental Theorem of Line integrals

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

This means, if $\vec{F} = \nabla f$, then the line integral is independent of path.

"$\vec{F}$ conservative $\Rightarrow$ Independent of path"

② "Independent of path $\Leftrightarrow$ $\int_C \vec{F} \cdot d\vec{r} = 0$ on every closed loop C"

③ "Independent of path $\Rightarrow$ Conservative"

④ "Conservative $\Rightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$"

⑤ "$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and D open, simply-connected $\Rightarrow$ Conservative"

<u>NOTE</u> ①, ②, and ③ combined give us

"conservative $\Leftrightarrow$ independent of path $\Leftrightarrow$ $\int_C \vec{F} \cdot d\vec{r} = 0$ on every closed loop C"

<u>NOTATION</u> $X \Rightarrow Y$ "If X, then Y"

$X \Leftrightarrow Y$ "X if and only if Y"