Lecture #12

16.2 Line Integrals

Our next goal is to learn to compute line integrals, specifically line integrals through vector fields (think of the vector field as doing work on a particle as it follows some trajectory in the vector field). First, however, we need to some other notions first.

We will first talk about line integrals with respect to arc length, then line integrals with respect to x, then with respect to y, then finally line integrals over vector fields. Each of these will require that we <u>parametrize curves</u> and understand how to <u>compute</u> them. The next page is a cheat sheet full of the formula's we will develop. The actual lecture starts on the page after.

(*) For each line integral below, we assume the curve C is parametrized by a function $\vec{r}(t)$, where $a \le t \le b$, and $\vec{r}(t) = \langle x(t), y(t) \rangle$. (*)

Line Integral with respect to Arc Length

$$\int_C f(x,y)\,ds = \int_{t=a}^{t=b} f(x(t),y(t)) \cdot \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

Line Integral with respect to x

$$\int_C f(x,y)\,dx = \int_{t=a}^{t=b} f(x(t),y(t)) \cdot x'(t)\,dt$$

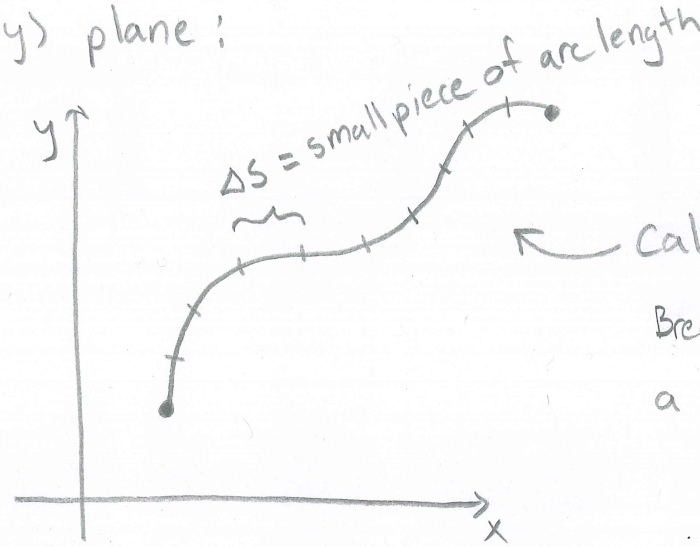
Line Integral with respect to y

$$\int_C f(x,y)\,dy = \int_{t=a}^{t=b} f(x(t),y(t))\,y'(t)\,dt$$

Line Integral over Vector Field

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\,ds = \int_{t=a}^{t=b} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

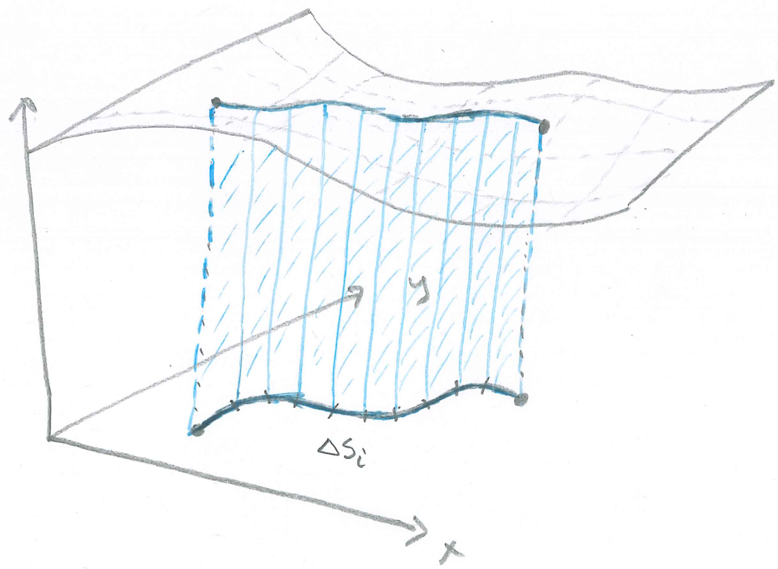
Say we have some nice function $f(x,y)$, and a curve in the (x,y) plane:



Call this curve C.

Break the curve into a bunch of pieces...

At each point (x,y) along this curve, $f(x,y)$ is some number.



If we wanted to compute the area of the "curtain" in **BLUE** above, then we would need to compute

$$\lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{f(x_i^*,y_i^*)}_{\text{height associated with}}\cdot\underbrace{\Delta s_i}_{\text{small piece of arc length}}$$

The limit of this becomes an integral:

$$\int_C f(x,y)\, ds$$

where $\underline{C}$ is the curve / trajectory and $\underline{ds}$ is an infinitesimally small piece of arc length.

QUESTION How do you compute such a thing?

First, recall arc length:

$$\text{arc length} = L = \int_C \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$ (see 10.2 in book)

Now, assume we have a parametrization of C:

$$C = \{ (x(t), y(t)) \text{ where } a \leq t \leq b \}$$

(or)

$$\begin{cases} x(t) \\ y(t) \end{cases}$$

(or)

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

Then, notice (NOT RIGOROUS!)

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{\frac{dx^2 + dy^2}{dx^2}} \cdot dx = \sqrt{dx^2 + dy^2}$$

and for $\begin{cases} x(t) \\ y(t) \end{cases}$ a parametrization, we have

$$dx = \frac{dx}{dt} \cdot dt = x'(t)\,dt$$

$$dy = \frac{dy}{dt} \cdot dt = y'(t)\,dt$$

so

$$\sqrt{dx^2 + dy^2} = \sqrt{dt^2\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

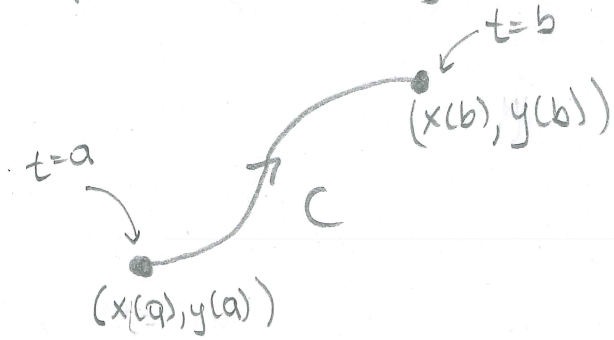
(See 10.2 in the text for a better explanation if you want.)

Then, $\displaystyle\int_C f(x,y)\,ds = \int_{t=a}^{t=b} f(x(t),y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt.$

<u>DEF</u> The "line integral with respect to arc length" is

$$\int_C f(x,y)\,ds = \int_{t=a}^{t=b} f(x(t),y(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2}\; dt$$

where C is parametrized by $\langle x(t), y(t)\rangle$, for $a \le t \le b$



So to begin, most of the time, our first challenge in computing line integrals is representing the curve C as a parametric equation in t.

EXAMPLE 1 Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of the unit circle.

STEP 1 Parametrize C!

Here we have a circle:

$$\left.\begin{aligned} x(t) &= r\cos t \\ y(t) &= r\sin t \end{aligned}\right\} \text{ want } r=1 \text{ since this is a unit circle.}$$

Notice, $(x(0), y(0)) = (1,0)$ and as t increases, $(x(t), y(t))$ moves counterclockwise.

⇒ If you want a refresher on how to parametrize circles (and ellipses), see the notes on the course website called "Parametrizing Circles and Ellipses" in the "Other Materials" section. (*)

Then, for our parametrization,

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \quad \text{for } \underline{0 \leq t \leq \pi}.$$

<u>STEP 2</u>: Plug and chug:

$$\int_C (2 + x^2 y)\, ds = \int_{t=0}^{t=\pi} \Big(2 + \overbrace{\cos^2 t}^{x(t)^2} \cdot \overbrace{\sin t}^{y(t)}\Big) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$x'(t) = -\sin t$

$y'(t) = \cos t$

$$= \int_0^{\pi} (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{\pi} (2 + \cos^2 t \sin t)\, dt$$

u-sub, careful! you get a negative! $u = \cos t$ $du = \underline{-}\sin t\, dt$.

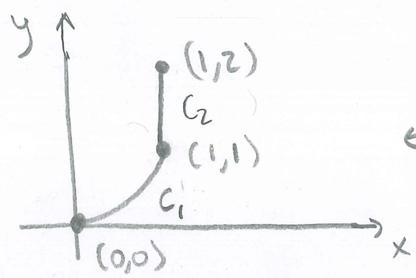
$$= \left[2t + \frac{-\cos^3 t}{3} \right]_0^{\pi}$$

$$= \left(2\pi + \frac{1}{3}\right) - \left(0 - \frac{1}{3}\right)$$

$$= \boxed{2\pi + \frac{2}{3}}$$

EXAMPLE 2 $\int_C 2x \, ds$, where C consists of the arc C_1 of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$, followed by the vertical line segment from $(1,1)$ to $(1,2)$.

STEP 1: DRAW!



← piecewise smooth
(continuous, smooth on "pieces")

$$\int_C 2x \, ds = \int_{C_1} 2x \, ds + \int_{C_2} 2x \, ds$$

STEP 2: Compute $\int_{C_1} 2x \, ds$.

Let $x(t)=t$. Then since $y=x^2$, $y(t)=t^2$.

And we have a parametrization!

$$\begin{cases} x(t)=t \\ y(t)=t^2 \end{cases} \quad \text{for } 0 \le t \le 1$$

Where do we start and end? start: $t=0$, get $(0,0)$
end: $t=1$, get $(1,1)$.

Then, plug and chug!

$$\int_{C_1} 2x\,ds = \int_{t=0}^{t=1} \underbrace{2t}_{x(t)} \sqrt{\underbrace{1^2}_{x'(t)=1} + \underbrace{(2t)^2}_{y'(t)=2t}}\,dt$$

$$= \int_0^1 2t\sqrt{1+4t^2}\,dt$$

u-sub: $u = 1+4t^2$

$du = 8t\,dt$, so $\frac{1}{4}du = 2t\,dt$

$$= \frac{1}{4}\int_{t=0}^{t=1} \sqrt{u}\,du$$

$$= \frac{1}{4}\left[\frac{2u^{3/2}}{3}\right]_{t=0}^{t=1} = \frac{1}{4}\left[\frac{2(1+4t^2)^{3/2}}{3}\right]_{t=0}^{t=1}$$

$$= \frac{1}{6}\left(5^{3/2}-1\right) = \boxed{\frac{5\sqrt{5}-1}{6}}$$

STEP 3: Compute $\int_{C_2} 2x\,ds$.

Notice, x is always 1 on the line segment. Then let $y(t)=t$...

$$\begin{cases} x(t)=1 \\ y(t)=t \end{cases} \quad \text{for } 1 \le t \le 2$$

Plug and chug! $\int_{C_2} 2x\,ds = \int_{t=1}^{t=2} \overset{(x(t)=1)}{2}\sqrt{\underbrace{0^2}_{x'(t)=0}+\underbrace{1^2}_{y'(t)=1}}\,dt = \int_1^2 2\,dt = \left[2t\right]_1^2 = \boxed{2}$

STEP 4:

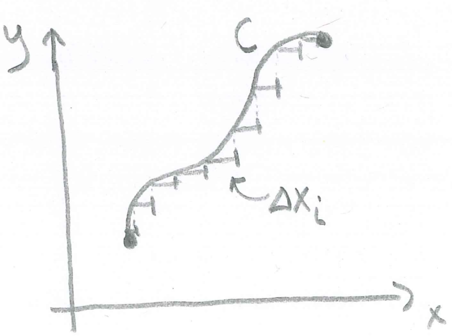
$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \boxed{\frac{5\sqrt{5}-1}{6} + 2}$$

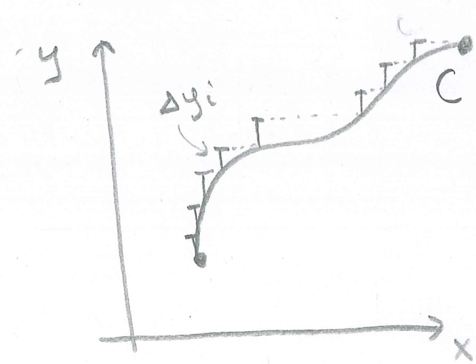
We can also define line integrals with respect to <u>x</u> and <u>y</u>. By this, we mean, instead of summing as follows:

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \cdot \Delta s_i ,$$

we want to sum with respect to changes in <u>x</u> or <u>y</u>:

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \cdot \Delta x_i \quad \underline{\text{OR}} \quad \sum_{i=1}^{n} f(x_i^*, y_i^*) \cdot \Delta y_i$$





Taking limits, we get two new line integrals:

$$\int_C f(x,y)\,dx \qquad\qquad \int_C f(x,y)\,dy$$

"line integral with respect to x" "line integral with respect to y"

QUESTION How do you compute these??

As before, we need a parametrization of the curve C.

Let $C \rightsquigarrow \begin{cases} x(t) \\ y(t) \end{cases}$ for $a \le t \le b$

Then

think "u-sub"

"$x(t) = t^2 + \cdots$

$dx = \underbrace{(2t + \cdots)}_{x'(t)} dt$"

DEF "line integral with respect to x"

$$\int_C f(x,y)\, dx = \int_{t=a}^{t=b} f(x(t), y(t))\, x'(t)\, dt$$

DEF "line integral with respect to y"

$$\int_C f(x,y)\, dy = \int_{t=a}^{t=b} f(x(t), y(t)) \cdot y'(t)\, dt$$

REMARK Sometimes, we will be interested in computing something like

$$\int_C P(x,y)\,dx + Q(x,y)\,dy.$$

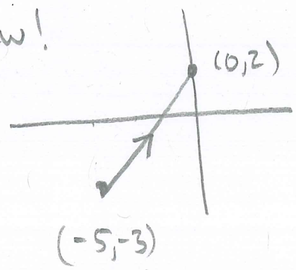
We do this by letting

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

and computing it by using the boxed equations above.

EXAMPLE 3 $\int_C y^2\,dx + x\,dy$, where C is the line segment from $(-5,-3)$ to $(0,2)$.

STEP 1: Draw!



STEP 2: Parametrize the line. It is of the form:

$$x(t) = at + b$$
$$y(t) = ct + d$$

for some a, b, c, d. Find them! Assume at $t=0$, we are at the starting point $(-5,-3)$, and at $t=1$, we are at the ending point $(0,2)$.

Then:

$$\boxed{x(0) = -5} \text{ and } \boxed{x(0) = a\cdot 0 + b}$$
$$\Rightarrow b = -5$$

And

$$\boxed{y(0) = -3} \text{ and } \boxed{y(0) = c\cdot 0 + d}$$
$$\Rightarrow d = -3.$$

Then, we know:

$$x(t) = at - 5$$
$$y(t) = ct - 3$$

To find a and c, use the other fact we know, at $t=1$, we are at $\boxed{(0,2)}$:

$$x(1) = 0 \quad \text{and} \quad x(1) = a \cdot 1 - 5$$

$$\Rightarrow 0 = a - 5 \Rightarrow \boxed{a=5}$$

$$y(1) = 2 \quad \text{and} \quad y(1) = c \cdot 1 - 3$$

$$\Rightarrow 2 = c - 3 \Rightarrow \boxed{c=5}$$

So,

$$\begin{cases} x(t) = 5t - 5 \\ y(t) = 5t - 3 \end{cases} \quad \text{for } 0 \le t \le 1$$

(Alternatively, you may remember from a previous course, depending on how it was taught:

Δx for 1 unit of time, "horizontal velocity"

$$x(t) = 5t - 5$$

starting point

$$y(t) = 5t - 3$$

Δy for 1 unit of time, "vertical velocity")

STEP 3: Plug and chug:

$$\int_C y^2 dx + x dy = \int_{t=0}^{t=1} \underbrace{(5t-3)^2}_{(y(t))^2} \cdot \underbrace{5}_{x'(t)=5} dt + \underbrace{(5t-5)}_{x(t)} \cdot \underbrace{5}_{y'(t)=5} dt$$

$$= \int_{t=0}^{t=1} \left((5t-3)^2 + (5t-5) \right) \cdot 5\,dt$$

$$= \int_{t=0}^{1} (25t^2 - 30t + 9 + 5t - 5) \cdot 5\,dt$$

$$= 5 \left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t \right]_{t=0}^{t=1}$$

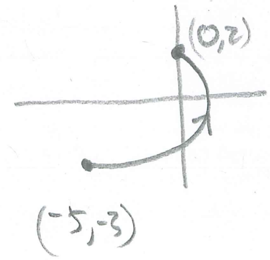
$$= 5 \left[\frac{25}{3} - \frac{25}{2} + 4 \right]$$

$$= 5 \left(\frac{50}{6} - \frac{75}{6} + \frac{24}{6} \right)$$

$$\boxed{= \frac{-5}{6}}$$

EXAMPLE 4 $\int_C y^2\,dx + x\,dy$, C is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

STEP 1: Draw!



STEP 2: Parametrize C. C is the curve $x = 4 - y^2$, so if we let $y = t$, then $x = 4 - t^2$

Then

$$\begin{cases} x(t) = 4 - t^2 \\ y(t) = t \end{cases} \quad \text{for} \quad -3 \le t \le 2.$$

And t goes from what to what? Notice, y starts at -3 and goes to 2 on the curve ...

<u>STEP 3</u>: Plug and chug!

$$\int_C y^2\,dx + x\,dy = \int_{t=-3}^{t=2} \overset{(y(t))^2}{(t^2)} \cdot \overset{x'(t)=-2t}{(-2t)}\,dt + \overset{x(t)}{(4-t^2)} \cdot \overset{y'(t)=1}{(1)}\,dt$$

$$= \int_{-3}^{2} \left[-2t^3 + (4-t^2)\right] dt$$

$$= \left[\frac{-2t^4}{4} + 4t - \frac{t^3}{3}\right]_{t=-3}^{t=2}$$

$$= \left(\left(\frac{-32}{4} + 8 - \frac{8}{3}\right) - \left(\frac{-81}{2} - 12 + 9\right)\right)$$

$$= \boxed{40\tfrac{5}{6}}$$

<u>Notice!</u> In the last two examples, the line integral was the same, except C was a different path between $(-5,-3)$ and $(0,2)$.

This means that in general, when computing line integrals, the value of the line integral is dependent on the path between two points!

However, there are special conditions which will cause the line integral to be independent of the path...

REMARK In addition, notice that the last two examples depended on the direction that the curves were parametrized, i.e. in both examples, we started at $(-5,-3)$ and went to $(0,2)$.

EXERCISE Work the last two examples with a reversed orientation/direction (start at $(0,2)$ and go to $(-5,-3)$.). You will find:

$$\int_{-C} f(x,y)\,dx = -\int_{C} f(x,y)\,dx$$

NOTATION for reversing the direction

And,

$$\int_{-C} f(x,y)\,dy = -\int_{C} f(x,y)\,dy.$$

EXERCISE Is the same true for $\int_C f(x,y)\,ds$?

Why or why not?

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Line Integrals in Space (w.r.t. arc length)

$$(*) \quad \int_C f(x,y,z)\,ds = \int_{t=a}^{t=b} f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

where C is parametrized by

$$\begin{cases} x(t) \\ y(t) \\ z(t) \end{cases} \quad \text{for } a \le t \le b.$$

EXERCISE Work EXAMPLE 5 and EXAMPLE 6 in the text (16.2)

Key: parametrizing a curve in 3D-space!

(EXAMPLE 6)

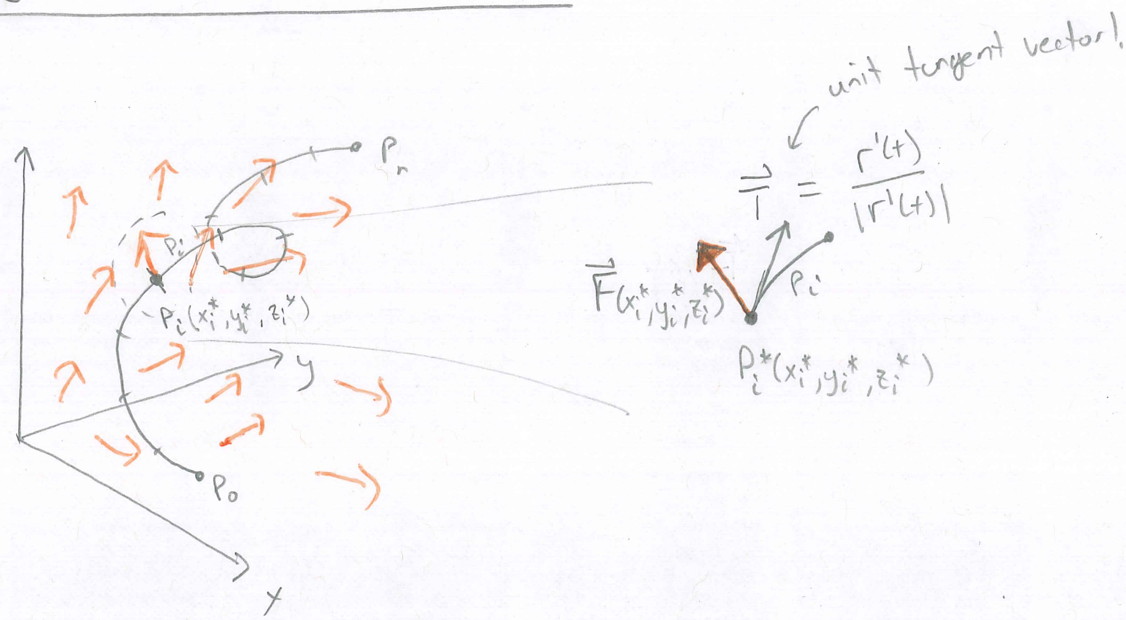
Line Integrals over Vector Fields

We move on to the last type of line integral we will study, a line integral over a vector field. At first, this will seem unrelated to what we just did, but it is actually very related... which we will see at the end of this section.

This part of the lecture notes starts on the next page. There is a video lecture on Canvas (Panopto) that goes with it.

EXERCISE Watch the video lecture!

Line integrals over Vector Fields



Question : How much work does the force field do on the particle?

REM $W = \vec{F} \cdot \vec{D}$ ← "displacement vector"

For us $\vec{D} \approx \underbrace{(\Delta s_i)}_{\text{small piece of arc length (length of } P_i \text{ above)}} \underbrace{\vec{T}(t)}_{}$ ← in the direction of the unit tangent vector.

So :

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
$\vec{T}(t)$ is in "x, y, z" also!

$$W \approx \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*) \Delta s_i$$

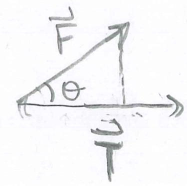
Taking a limit as our pieces get smaller and smaller..

$$W = \int_C \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\, ds$$

$$= \int_C \vec{F} \cdot \vec{T}\, ds$$

"Work is the line integral with respect to arc length of the <u>tangential</u> <u>component</u> of the force"

(*) [<u>rem</u> $\vec{F} \cdot \vec{T} = |\vec{F}| \cdot |\vec{T}| \cos\theta$



] (*)

If the curve C is given by a vector equation

$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$, then

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}, \quad \text{so}$$

think <u>u-sub</u> with $\vec{r}(t)$!

$$W = \int \left[\vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right] \cdot |\vec{r}\,'(t)|\, dt = \int_{t=a}^{t=b} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

DEF "The line integral of F along C"

$$\int_C \overset{\text{"abbreviation"}}{\vec{F}\cdot d\vec{r}} = \underbrace{\int_{t=a}^{t=b} F(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt}_{\text{(this is the one for computations most of the time!)}} = \int_C \vec{F}\cdot\vec{T}\,ds$$

Here, we assume $\vec{F}$ is a continuous vector field defined on a smooth curve $\vec{r}(t)$, $a \le t \le b$.

EXAMPLE 5 Find the work done by $\overset{\text{the force field}}{\vec{F}(x,y)} = x^2\,\hat{\imath} - xy\,\hat{\jmath}$ in moving a particle along the quarter circle

$$\vec{r}(t) = \langle \underset{x(t)}{\cos t}, \underset{y(t)}{\sin t} \rangle, \quad t = 0 \text{ to } t = \pi/2.$$

$$\vec{F}(\vec{r}(t)) = \underbrace{\cos^2(t)}_{(x(t))^2}\,\hat{\imath} - \underbrace{\cos t\sin t}_{x(t)\cdot y(t)}\,\hat{\jmath}$$

$$\vec{r}\,'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}$$

Then,

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{t=\pi/2} \left(\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\right)dt = \int_{t=0}^{t=\pi/2} \left(-2\cos^2 t\sin t\right)dt$$

$$= \frac{2\cos^3 t}{3}\Big]_0^{\pi/2}$$

$$\boxed{= -\frac{2}{3}}$$

Exercise · Compute the example above with a <u>reversed</u> <u>orientation</u>.

· Does the answer make sense?

· Why does orientation matter here even though we can write this as an integral with respect to arc length?

REMARK What is the connection between scalar line integrals and integrals over vector fields?

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, each functions of (x,y,z), $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$= \int_a^b (P\hat{\imath} + Q\hat{\jmath} + Rk) \cdot (x'(t)\,\hat{\imath} + y'(t)\,\hat{\jmath} + z'(t)\,\hat{k})\, dt$$

$$= \int_a^b P(x(t), y(t), z(t)) \cdot x'(t)\, dt + Q(x(t), y(t), z(t)) \cdot y'(t)\, dt + R(x(t), y(t), z(t)) \cdot z'(t)\, dt$$

$$= \int_C P(x,y,z)\, dx + Q(x,y,z)\, dy + R(x,y,z)\, dz$$

* exercise understand each step!