16.3 The Fundamental Theorem for Line Integrals

REM Fundamental Theorem of Calculus, part 2

$$\int_a^b F'(x)\,dx = F(b) - F(a),$$

where F' is continuous on $[a,b]$.

Exercise Write the integral above as a line integral through a 1-dimensional vector field. Hint: Let $F'(x)$ be the component function for a vector field (with 1-dim).

THM (Fundamental Theorem for Line Integrals)

Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

REMARK • We should notice a few things: first, if the vector field is the gradient of a surface, then the path between endpoints does not matter, only the endpoints!

- Second, orientation matters. It matters whether you start at $\vec{r}(a)$ and go to $\vec{r}(b)$ (as in the theorem), or start at $\vec{r}(a)$ and go to $\vec{r}(b)$.

- Lastly, recall that conservative vector fields are vector fields that are the gradient of a surface. So, you can read the theorem as a statement about conservative vector fields. In fact, the theorem says that if $\vec{F} = \nabla f$, then the line integral will be the net change in f from $\vec{r}(a)$ to $\vec{r}(b)$.

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proof of THM: Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \quad \text{, by definition}$$

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \Big|_{\vec{r}(t)} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}_{= \langle x'(t), y'(t), z'(t) \rangle} dt$$

$\vec{r}(t)$: $x = x(t),\ y = y(t),\ z = z(t)$

$$= \int_a^b \left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} \right) dt$$

Chain rule!

$$= \int_a^b \frac{d}{dt} \left(f(\vec{r}(t)) \right) dt$$

Fundamental Theorem of Calculus!

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

Exercise: understand each line! Be able to reproduce this proof.

EXAMPLE 1 Find the work done by the gravitational field

$$\vec{F}(\vec{x}) = \frac{-mMG}{|\vec{x}|^3}\vec{x}$$

in moving a particle with mass m from the point (3,4,12) to the point (2,2,0).

Remember! In the previous notes, one of the exercises was to determine which of the vector fields is conservative. The vector field above is conservative! In doing the exercise, you should have determined that

$$\vec{F}(\vec{x}) = \nabla f,$$

where

$$f = \frac{mMG}{\sqrt{x^2+y^2+z^2}}.$$

(Find this by integrating!)

Knowing this, we can compute the work done quickly!

$$W = \int_C \vec{F}\,d\vec{r} = \int_C \nabla f \cdot d\vec{r} \xrightarrow{\text{F.T. of Line Integrals}} = f(2,2,0) - f(3,4,12)$$

$$= \frac{mMG}{\sqrt{8}} - \frac{mMG}{\sqrt{169}}$$

$$\boxed{= mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)}$$

REMARK Remember EXAMPLES 3 and 4 from the previous notes. In general, for curves C_1 and C_2 with the same endpoints, but different paths,

$$\int_{C_1} \vec{F}\cdot d\vec{r} \neq \int_{C_2} \vec{F}\cdot d\vec{r}$$

But, the Fundamental theorem tells us that for conservative vector fields,

$$\int_{C_1} \nabla f\cdot d\vec{r} = \int_{C_2} \nabla f\cdot d\vec{r}$$

where C_1 and C_2 are as before.

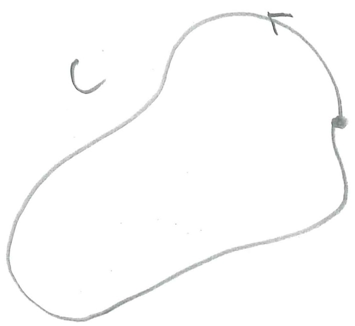
DEF If $\vec{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ for any two paths C_1 and C_2 in D that have the same initial points and the same terminal points. (Note: this does not say $\vec{F}$ is conservative... yet.)

REMARK Line integrals of conservative vector fields are independent of path. One may ask, are all path independent line integrals line integrals of conservative vector fields?

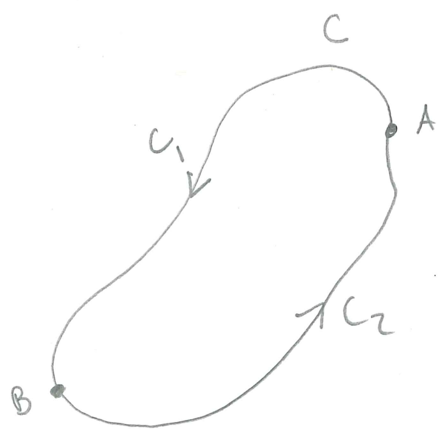
Key question!

We will try to answer this!

<u>DEF</u> A curve C is called <u>closed</u> if its terminal point is the same as its initial point, i.e. $\vec{r}(a) = \vec{r}(b)$.



<u>Notice</u>, we can pick two points on a closed curve C, and think of C as the composition of two curves C_1 and C_2:



Notice: $\cdot$ C_1 goes from A to B
$\cdot$ C_2 goes from B to A.

Then, if we have a line integral that is <u>independent of path</u>, we can compute a line integral over C:

$$\int_C \vec{F}\,d\vec{r} = \int_{C_1} \vec{F}\,d\vec{r} + \int_{C_2} \vec{F}\,dr$$

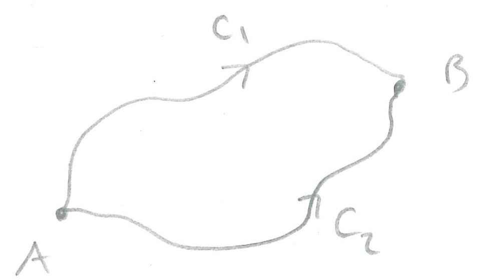
$$= \int_{C_1} \vec{F}\,d\vec{r} - \int_{-C_2} \vec{F}\,dr$$

Exercise: what allows us to take this step?

$$= (\vec{F}(b) - \vec{F}(a)) - (\vec{F}(b) - \vec{F}(a))$$

$$= 0.$$

Conversely, if $\int_C \vec{F}\cdot d\vec{r} = 0$ for any closed path in D, we can show the integral is path independent. Pick any two points A and B, and pick any two paths from A to B.



Notice, C_1 and $-C_2$ composed gives us a closed loop! Call that loop C. Then

$$0 = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\,d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\,d\vec{r}.$$

Thus, $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\,d\vec{r}$.

So, we have a theorem!

THM $\int_C \vec{F} \cdot d\vec{r}$ is independent of path if and only if

$\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C. (in D, where $\vec{F}$ is cont. in D).

We can use this to show:

THM Suppose $\vec{F}$ is a vector field that is continuous on an $\boxed{\text{open, connected}}$ (see below) region D. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \vec{F}$.

End of Friday's (7/20) Lecture!

DEF By open above, we mean that for each point P in the domain, there is a small disk with center P that lies completely in D.

"It doesn't have a boundary"

or, think open intervals: (a,b) (don't contain a or b!)

DEF By <u>connected</u>, we mean that any two points in D can be joined by a path in D.

(**REMARK** Later, this becomes the definition of "path-connected," and in general, connected doesn't mean this. But, for nice spaces, like $\mathbb{R}^n$, it does!)

Exercise Read the proof of the THM above in the text (starting on p. 1129).

Sketch of proof:
- Assume $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.
- Fix a point (a,b) in D.
- Define $f(x,y) = \displaystyle\int_{(a,b)}^{(x,y)} \vec{F} \cdot d\vec{r}$.

Why do this? Notice that $f(a,b) = 0$, and $f(x,y)$ will be the <u>net</u> change in the function $f(x,y) - f(a,b) = f(x,y)$.

Why fix any (a,b)? If I fix $(\tilde{a}, \tilde{b})$, is $f(x,y)$ different? Yes, it is! But, since we want $\vec{F} = \nabla f$, and the ∇f requires taking derivatives, there will be lots of f's give the same gradient. (Rem: $f'(x) = x$, then $f(x) = \frac{x^2}{2} + C$, which is a family of functions! Same goes for higher dimensions!)

Next, notice no path between (a,b) and (x,y)

is specified. This is because we are assuming

$\int_C \vec{F} \cdot d\vec{r}$ is independent of path -- this also tells us

that $f(x,y)$ is <u>well-defined</u>. No matter what path we pick

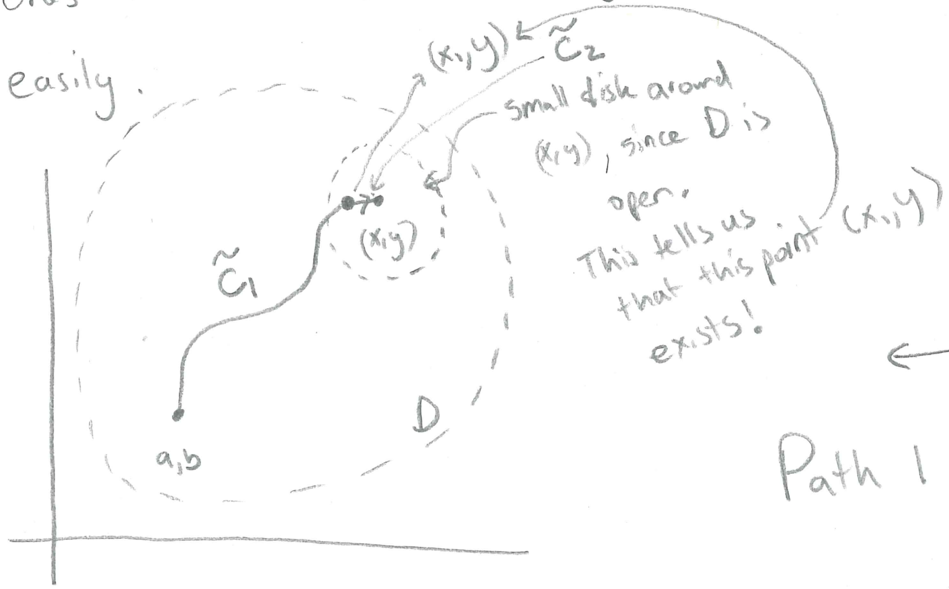
between (a,b) and (x,y), $\int_C \vec{F} \cdot d\vec{r}$ will give us the

<u>same</u> number! So, lets pick 2 convenient paths...

ones that will give us a way to take partial derivatives

easily.

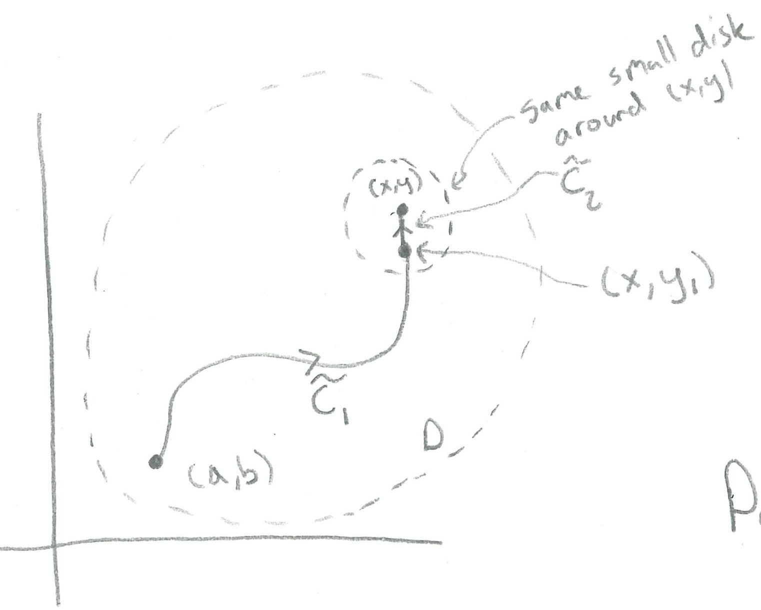
CASE 1



<u>Key</u>: (x_1,y) and (x,y) have the same y-coordinate!

Path 1: $C_1 = \tilde{C}_1 + \tilde{C}_2$

CASE 2



<u>Key</u>: (x,y_1) and (x,y) have the same x-coordinate!

Path 2: $C_2 = \tilde{C}_1 + \tilde{C}_2$

In Case 1, $f(x,y) = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$

$$= \underbrace{\int_{(a,b)}^{(x_1,y)} \vec{F} \cdot d\vec{r}}_{\text{not dependent on } x} + \underbrace{\int_{C_2} \vec{F} \cdot d\vec{r}}_{\text{depends on } x}$$

so, $\frac{\partial}{\partial x} f(x,y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \vec{F} \cdot d\vec{r}$.

Now, write $\vec{F} = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$. Parametrize C_2. Notice y is constant, so $\underline{dy = 0 \cdot dt}$, and we can use $t = x$ for $x_1 \le t \le x$ as the parametrization. Then

$$\left[\begin{array}{l} dx = x'(t)\,dt \\ \quad = dt \end{array}\right]$$

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} \int_{C_2} \vec{F}\, d\vec{r} = \frac{\partial}{\partial x} \int_{C_2} P(x,y)\, dx + Q(x,y)\, dy$$

$$= \frac{\partial}{\partial x} \left(\int_{x_1}^{x} P(t,y)\, dt \right)$$

$$= P(x,y) \quad \text{by FTC, part 1.}$$

So, $\frac{\partial f}{\partial x} = P(x,y)$, as desired! We can use Case 2 to show $\frac{\partial f}{\partial y} = Q(x,y)$, completing the proof!

So, that answers the question from the remark on page 4 of these notes, but it leads to other questions.

Since being conservative is such a <u>strong</u> condition, it would be nice to know <u>when</u> vector fields are conservative ...

<u>REM</u> If $f(x,y)$ is smooth, then

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}}.$$

<u>REMARK</u> If a vector field is conservative, then

$$\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath}$$
$$= \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

for some $f(x,y)$.

Then, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ since $\frac{\partial P}{\partial y} = \frac{\partial f}{\partial y \partial x} = \frac{\partial f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$.

<u>THM</u> If $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$ is a conservative vector field, where P and Q have continuous first order partial derivatives on a Domain D, then throughout D, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

["If conservative, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$"]

Wouldn't it be nice if the converse were true?

Unfortunately, it isn't! But with an added assumption, we will be able to prove, once we have Green's Theorem:

THM Let $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ be a vector field on an open $\boxed{\text{simply-connected}}$ region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D. Then F is conservative.

["If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ <u>and</u> D is open, simply connected, then F is conservative"]

The <u>key</u> here is that the domain must be <u>simply-connected</u>!

But what does this mean? Great question.

DEF We say a <u>simple</u> <u>curve</u> is a curve that does not intersect itself anywhere between endpoints.

simple, closed — Simple, not closed — not simple, not closed — not simple, closed.

DEF A simply-connected region is a connected region such that every simple closed curve in D encloses only points that are in D.

(Assume each set is open)

simply-connected region

not simply-connected!

not connected, so not simply-connected. (each piece is simply-connected)

EXAMPLE 2 Is $F(x,y) = (x-y)\,\hat{\imath} + (x-2)\,\hat{\jmath}$ conservative?

Notice, letting $P(x,y) = x-y$, $Q(x,y) = x-2$, we see

$$\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1$$

and $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$. Then, $\vec{F}$ is not conservative.

(Rem If $\vec{F}$ is conservative $\Longrightarrow$ then $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

This means If $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x} \Longrightarrow \vec{F}$ is not conservative.

contrapositive!

EXAMPLE 3 Is $\vec{F}(x,y) = (3+2xy)\hat{\imath} + (x^2-3y^2)\hat{\jmath}$ conservative?

Let $P = 3+2xy$
$Q = x^2-3y^2$

Then, $\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$.

This does not mean $\vec{F}$ is conservative! We must first check that $\vec{F}$ is defined on a simply connected region. Since the domain of $\vec{F}$ is all of $\mathbb{R}^2$, we must ask if $\mathbb{R}^2$ is simply connected.

Since it is, we can say $\vec{F}$ is conservative!

Exercise Read and work EXAMPLES 4 and 5 in the text. These examples show the strengths & limitations of the theorems we have.

Exercise

- Is $\vec{F}(x,y) = -y\hat{\imath} + x\hat{\jmath}$ a conservative vector field?
- Is $\vec{F}(x,y) = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$ a conservative vector field? (Do true/false first!)

Exercise (True or False)

1. Let C be any closed path on a conservative vector field. Then $\int_C \vec{F} \cdot d\vec{r} = 0$.

2. Line integrals on vector fields depend only on the initial point and terminal point of a curve.

3. A line integral is independent of path on a domain D if and only if the vector field on D is conservative.

4. Let $\vec{F}$ be a vector field such that it is smooth on all of $\mathbb{R}^2$. If $\vec{F}(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$, and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $\vec{F}$ is conservative.

5. Let $\vec{F}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$ on an open, connected domain D. If $\vec{F}$ is conservative, then $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$.

6. If line integrals over a vector field $\vec{F}$ are independent of path on a domain D, and $\vec{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

7. Suppose $\int_C \vec{F} \cdot d\vec{r} = 0$ on every closed loop in the domain $D = \{(x,y) : \frac{1}{2} < x^2+y^2 < 1\}$. Then $\vec{F}$ is a conservative vector field on D.

8. Suppose $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on the open, connected region D when $\vec{F}$ is defined and smooth. Then $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$, where C_1 and C_2 are curves in D with the same endpoint.

9. Suppose $\vec{F}(x,y) = M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath}$ where $\vec{F}$ is smooth on a simply connected region D and $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$. Then on any closed loop C in D,

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$

10. Suppose $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$ is a differentiable vector field on an open, connected region D and $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$. Then $\vec{F}$ is not conservative on D, but $\vec{F}$ could be on a slightly larger region $\tilde{D}$ that contains D.