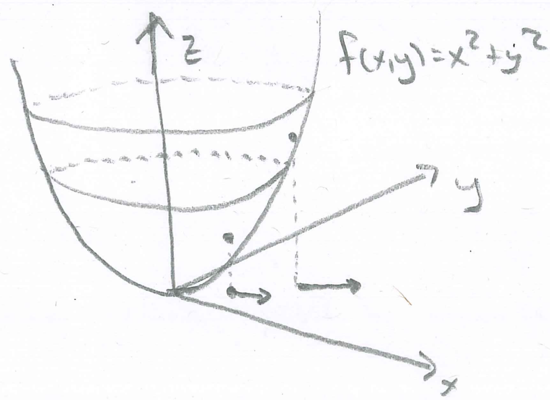
Lecture #11

16.1 Vector Fields

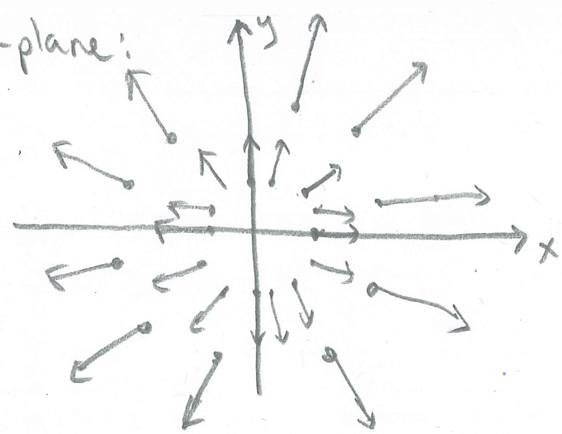
REM $\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$.

We can use the gradient to assign a vector to each point (x,y) in the xy-plane. For example, let $f(x,y) = x^2+y^2$, then $\nabla f(x,y) = \langle 2x, 2y \rangle$ and at each (x,y), we can attach a vector $\langle 2x, 2y \rangle$.

PIC



In the xy-plane:



$\nabla f = \langle 2x, 2y \rangle$

NOTICE : The gradient "vector field" always points in the direction that $f(x,y)$ increases the fastest!

ALSO Don't believe pictures! Plot some points of ∇f on your own, until you believe the picture ... (!)

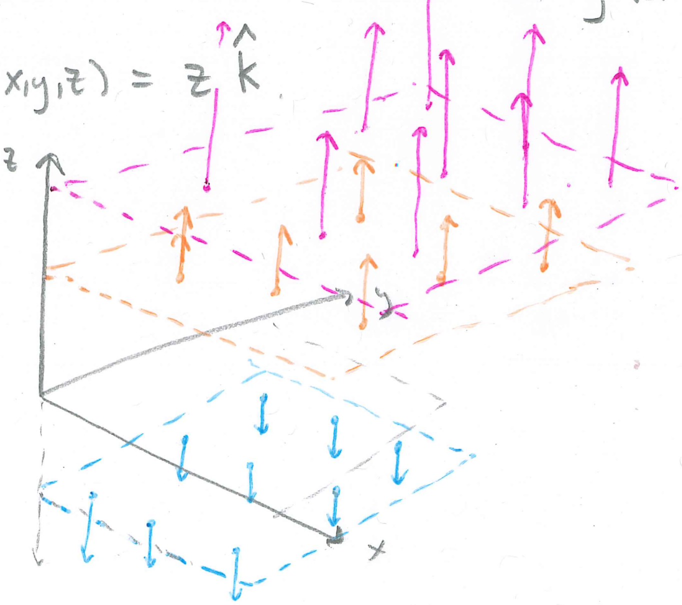
DEF Let D be a set in $\mathbb{R}^2$ (a plane region). A <u>vector field on D in $\mathbb{R}^2$</u> is a function $\vec{F}$ that assigns to each point (x,y) in D a two-dimensional vector $\vec{F}(x,y)$.

DEF Let E be a subset of $\mathbb{R}^3$. A <u>vector field on E in $\mathbb{R}^3$</u> is a function $\vec{F}$ that assigns to each point (x,y,z) in E a three-dimensional vector $\vec{F}(x,y,z)$.

REMARK We often write $\vec{F}(x,y)$ as $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$, and call $P(x,y)$ and $Q(x,y)$ <u>component functions</u>. Notice that $P(x,y)$ and $Q(x,y)$ are <u>scalar functions</u>. We sometimes call these <u>scalar fields</u>. (e.g. $P(x,y)$ assigns a scalar to each point instead of a vector.)

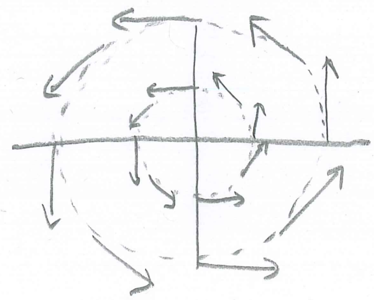
EXAMPLE 1 Sketch the vector field on $\mathbb{R}^3$ given by

$$F(x,y,z) = z\,\hat{k}.$$



EXAMPLE 2 Sketch the vector field on $\mathbb{R}^2$ defined by

$$\vec{F}(x,y) = -y\,\hat{\imath} + x\,\hat{\jmath}$$



(Make a table if it makes it easier for you!)

EXERCISE Do you recognize this vector field as something you have seen before? Why are there circles drawn on this vector field?

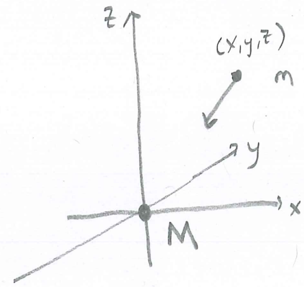
EXAMPLE 3 (Gravitational Field)

RECALL (from Physics) The gravitational force between two objects is given by the equation

$$F = \frac{mMG}{r^2}$$

where m is the mass of one of the objects, M the mass of the other, G the gravitational constant, and r the distance between the two objects. We can write a vector version of this equation!

Let M be the mass of an object at the origin. Then, an object of mass m located at an arbitrary point (x,y,z) in 3D-space experiences a pull towards the mass M at the origin, a gravitational pull from M.



The magnitude of this force must be equal to $\frac{mMG}{r^2}$ where r is the distance to the origin:

$$r = \sqrt{x^2+y^2+z^2}.$$

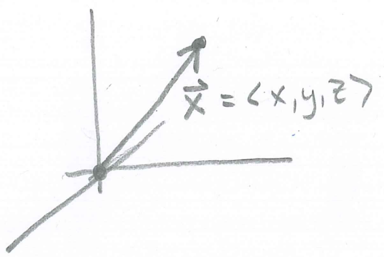
Notice that if $\vec{x} = \langle x,y,z \rangle$ is the vector pointing to (x,y,z), then

$$r = \sqrt{x^2+y^2+z^2} = |\vec{x}|,$$

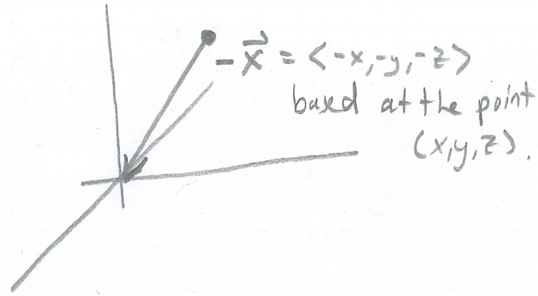
so we can rewrite the magnitude as $\frac{mMG}{|\vec{x}|^2}$.

This gives us the magnitude of the force, but <u>not</u> the direction. What direction should it point? Towards the origin!

What vector points in the direction of the origin? $-\vec{x}$!



and



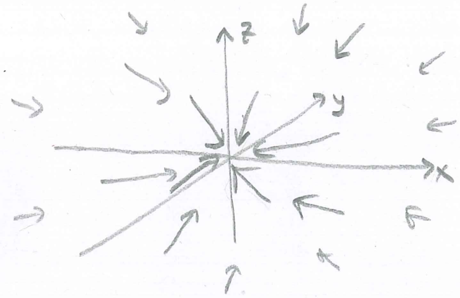
Now, we need to make $\vec{F}(x,y,z)$ a vector that points in the direction of the origin, with magnitude $\frac{mMG}{|\vec{x}|^2}$. In other words, take a unit vector pointing towards the origin, $\frac{-\vec{x}}{|-\vec{x}|} = \frac{-\vec{x}}{|\vec{x}|}$, and multiply it by the scalar $\frac{mMG}{|\vec{x}|^2}$:

$$\vec{F}(x,y,z) = \underbrace{\frac{mMG}{|\vec{x}|^2} \cdot \frac{-\vec{x}}{|\vec{x}|}}_{\text{vector in direction of } -\vec{x} \text{ with magnitude } \frac{mMG}{|\vec{x}|^2}.}$$

Make sure you understand this!

$$\boxed{\vec{F}(x,y,z) = \frac{-mMG}{|\vec{x}|^3} \cdot \vec{x}}$$

This vector field looks like:



- The further you are from the origin, the smaller the vector.
- All of the vectors point towards the origin.

What does it describe? Put the mass m at any point in the vector field (force field) and the vector at that location will point in the direction of the gravitational pull. How hard it is pulling is given by the magnitude of the vector.

We can break $\vec{F}(x,y,z)$ into components to see this a little better.

$$\vec{F}(x,y,z) = \frac{-mMG}{|\vec{x}|^3}\vec{x}$$

$$= \frac{-mMG}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\langle x,y,z\rangle$$

$$= \frac{-mMG}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$$

$$= \left\langle \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

EXERCISE Work through EXAMPLE 5 in 16.1 (Electric Fields!)

REMARK Fields of the form above are important! At least the form of the field is important. It will recur a few times in the course.

DEF We say a vector field $\vec{F}$ is a conservative vector field if it is the gradient of some scalar function; i.e. there exist some $f(x,y)$ such that $\nabla f = \vec{F}$.

REMARK Any "gradient vector field" is a conservative vector field, by definition!

EXERCISE Of the examples in the notes, which are conservative vector fields? Which are not? How can you tell?