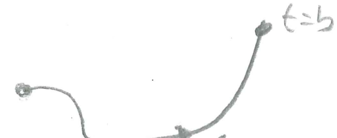
Lecture #11

RECALL

$$\int_C f(x,y)\,ds = \int_{t=a}^{t=b} f(x(t),\, y(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

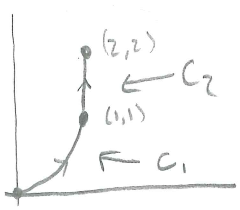
C is parametrized by $\begin{cases} x(t) \\ y(t) \end{cases}$



"line integral with respect to arc length"

EXAMPLE 2

$\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$, followed by the vertical line segment from $(1,1)$ to $(2,2)$.



← piecewise smooth (continuous, smooth on "pieces")

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

Compute integral over C_1 first:

$x(t) = t$

$y(t) = x(t)^2 = t^2$ $\Rightarrow$ $\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$, $t=0$ to $t=1$

where do you start/end?

Then,

$$\int_{C_1} 2x\,ds = \int_{t=0}^{t=1} (2t)\sqrt{1^2 + (2t)^2}\,dt = \int_{t=0}^{t=1} 2t\sqrt{1+4t^2}\,dt$$

u-sub: $u = 1+4t^2$

$du = 8t\,dt \rightsquigarrow \left(\frac{1}{4}du = 2t\,dt\right)$

$$= \frac{1}{4}\int_{t=0}^{t=1} \sqrt{u}\cdot du = \frac{1}{4}\left[\frac{2u^{\frac{3}{2}}}{3}\right]_{t=0}^{t=1} = \frac{1}{4}\left[\frac{2(1+4t^2)^{3/2}}{3}\right]_{t=0}^{t=1}$$

$$= \frac{1}{6}\left(5^{3/2} - 1\right)$$

$$\boxed{= \frac{5\sqrt{5}-1}{6}}$$

For C_2:

$x(t) = 1$

$y(t) = t$, $\underline{t=1 \text{ to } t=2}$ (where do you start / end)

$$\int_{C_2} 2x\,ds = \int_{t=1}^{t=2} 2\sqrt{0^2+1^2}\,dt = \int_{t=1}^{t=2} 1\,dt = \boxed{2}$$

Then

$$\int_C 2x\,ds = \boxed{\frac{5\sqrt{5}-1}{6} + 2}$$

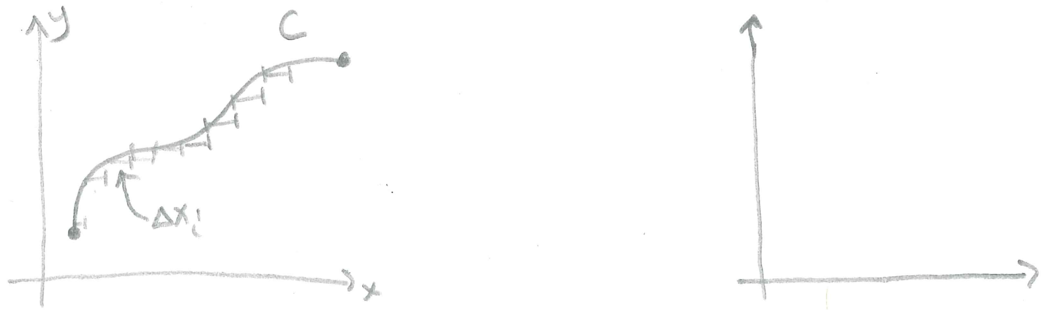
We can also define line integrals with respect to $\underline{x}$ and $\underline{y}$, instead of w.r.t. arc length.

By this, we mean, instead of summing as follows:

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \cdot \Delta s_i \, ,$$

we want to sum with respect to changes in $\underline{x}$ or $\underline{y}$:

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \cdot \Delta x_i \quad \text{or} \quad \sum_{i=1}^{n} f(x_i^*, y_i^*) \cdot \Delta y_i$$



Taking limits, we get two new line integrals:

$$\int_C f(x,y)\,dx \qquad\qquad \int_C f(x,y)\,dy$$

"line integral with respect to x" "line integral with respect to y"

To compute, we need a parametrization of the curve C.

Let

$$C \rightsquigarrow \begin{cases} x(t) \\ y(t) \end{cases} \quad \text{from } t=a \text{ to } t=b.$$

"$x(t) = t^2 + \ldots$"

$dx = \underbrace{(2t + \ldots)}_{\boxed{x'(t)}} dt$

think "u-sub"

Then:

$$\boxed{\begin{gathered} \int_{t=a}^{t=b} f(x(t), y(t)) \cdot x'(t)\, dt \\ \text{or} \\ \int_{t=a}^{t=b} f(x(t), y(t)) \cdot y'(t)\, dt \end{gathered}} \quad (*)$$

with respect to x

with respect to y.

<u>REMARK</u> Sometimes, we will be interested in computing something like

$$\int_C \big(P(x,y)\,dx + Q(x,y)\,dy\big).$$

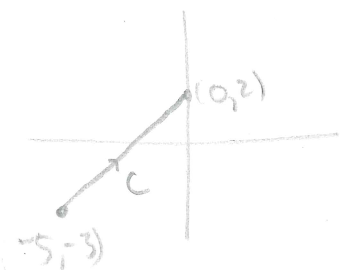
We do this by letting:

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

and computing using the boxed equations above.

EXAMPLE 3

$$\int_C y^2\,dx + x\,dy \,,$$ where C is the line segment from $(-5,-3)$ to $(0,2)$.



Parametrize the line:

Δx for 1 unit of time, "horizontal velocity"

$$x(t) = 5t - 5$$ ← starting point

$$y(t) = 5t - 3$$

Δy for 1 unit of time, "vertical velocity"

Then, the curve goes from $t=0$ to $t=1$.

$$\int_C y^2\,dx + x\,dy = \int_{t=0}^{t=1} \Big[\overbrace{(5t-3)^2}^{(y(t))^2}\cdot \overbrace{5}^{x'(t)}\,dt + \overbrace{(5t-5)}^{x(t)}\cdot \overbrace{5}^{y'(t)}\,dt\Big]$$

$$= \int_{t=0}^{t=1} \Big((5t-3)^2 + (5t-5)\Big)\cdot 5\,dt$$

$$= \int_{t=0}^{t=1} (25t^2 - 30t + 9 + 5t - 5)\,5\,dt$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_{t=0}^{t=1}$$

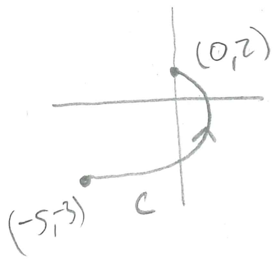
$$= 5\left[\frac{25}{3} - \frac{25}{2} + 4\right] = 5\left(\frac{50}{6} - \frac{75}{6} + \frac{24}{6}\right)$$

$$\boxed{= \frac{-5}{6}}$$

EXAMPLE 4

$$\int_C y^2 dx + x dy \quad , \qquad C \text{ is the arc of the parabola } x = 4 - y^2$$

from $(-5,-3)$ to $(0,2)$.



parametrize C:

if $y = t$,

then $x = 4 - t^2$.

So $\Rightarrow$ $\left.\begin{array}{l} x(t) = 4 - t^2 \\ y(t) = t \end{array}\right\}$ $t = -3$ to $t = 2$.

Then,

$$\int_C y^2 dx + x dy = \int_{t=-3}^{t=2} \overbrace{(t^2)}^{(y(t))^2} \cdot \overbrace{(-2t)}^{x'(t)} \cdot dt + \overbrace{(4-t^2)}^{x(t)} \cdot \overbrace{(1)}^{y'(t)} dt$$

$$= \int_{t=-3}^{t=2} (-2t^3) + (4 - t^2)\, dt$$

$$= \left[\frac{-2t^4}{4} + 4t - \frac{t^3}{3} \right]_{t=-3}^{t=2}$$

$$= \left(\left(\frac{-32}{4} + 8 - \frac{8}{3} \right) - \left(\frac{-81}{2} - 12 + 9 \right) \right)$$

$$\boxed{= 40 \tfrac{5}{6}}$$

Notice! In the last two examples, the line integral was the same, except that C was a different path between $(-5,-3)$ and $(0,2)$.

This means that in general, when computing line integrals, the value of the line integral is dependent on the path between two points!

However, there are special conditions which will cause the line integral to be independent of the path ...

Notice! The answers above were also dependent on orientation, i.e. in both examples, we started at $(-5,-3)$ and went to $(0,2)$.

Exercise Work the last two examples with a reversed orientation (start at $(0,2)$ and go to $(-5,-3)$).

You should find:

reversed orientation!

$$\int_{-C} f(x,y)\,dx = -\int_{C} f(x,y)\,dx$$

And,

$$\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy.$$

Exercise Is the same true for $\int_C f(x,y)\,ds$?

Why or why not?

Line integrals in Space

$$(*) \quad \boxed{\int_C f(x,y,z)\,ds = \int_{t=a}^{t=b} f(x(t), y(t), z(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt}$$

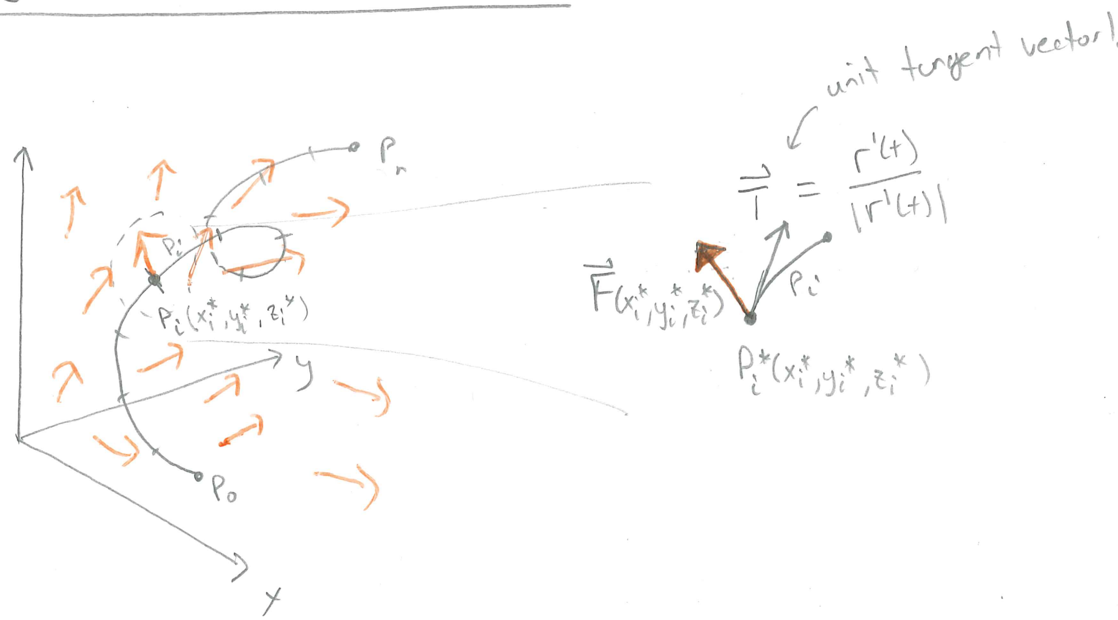
C is parametrized as follows

$$\begin{cases} x(t) \\ y(t) \end{cases}, \quad t=a \text{ to } t=b.$$

Exercise Work EXAMPLE 5 and EXAMPLE 6 in the text (16.2)

(Key: parametrizing a curve in 3D-space!)
(EXAMPLE 6)

Line integrals over Vector Fields



Question: How much work does the force field do on the particle?

REM $W = \vec{F} \cdot \vec{D}$ ← "displacement vector"

For us $\vec{D} \approx \underbrace{(\Delta s_i)}_{\text{small piece of arc length (length of } P_i \text{ above)}} \underbrace{\vec{T}(t)}$ ← in the direction of the unit tangent vector.

So:

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
$\vec{T}(t)$ is in "x, y, z" also!

$$W \approx \sum_{i=1}^{n} \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*) \Delta s_i$$

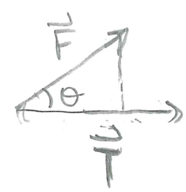
Taking a limit as our pieces get smaller and smaller..

$$W = \int_C \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\, ds$$

$$= \int_C \vec{F} \cdot \vec{T}\, ds$$

"Work is the line integral with respect to arc length of the <u>tangential</u> <u>component</u> of the force"

(*) [<u>rem</u> $\vec{F} \cdot \vec{T} = |\vec{F}| \cdot |\vec{T}| \cos\theta$



] (*)

If the curve C is given by a vector equation

$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$, then

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}, \quad \text{so}$$

think <u>u-sub</u> with $\vec{r}(t)$!

$$W = \int \left[\vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right] \cdot |\vec{r}\,'(t)|\, dt = \int_{t=a}^{t=b} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

DEF "The line integral of F along C"

"abbreviation"

$$\int_C \vec{F} \cdot d\vec{r} = \underbrace{\int_{t=a}^{t=b} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}_{} = \int_C \vec{F} \cdot \vec{T}\, ds$$

(this is the one for computations most of the time!)

Here, we assume $\vec{F}$ is a continuous vector field defined on a smooth curve $\vec{r}(t)$, $a \le t \le b$.

EXAMPLES Find the work done by the force field $\vec{F}(x,y) = x^2\,\hat{i} - xy\,\hat{j}$ in moving a particle along the quarter circle

$$\vec{r}(t) = \langle \underset{x(t)}{\cos t}, \underset{y(t)}{\sin t} \rangle, \quad t = 0 \text{ to } t = \pi/2.$$

$$\vec{F}(\vec{r}(t)) = \underbrace{\cos^2(t)}_{(x(t))^2}\,\hat{i} - \underbrace{\cos t \sin t}_{x(t)\cdot y(t)}\,\hat{j}$$

$$\vec{r}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

Then,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=\pi/2} \left(\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\right) dt = \int_{t=0}^{t=\pi/2} \left(-2\cos^2 t \sin t\right) dt$$

$$= \frac{2\cos^3 t}{3}\Big]_0^{\pi/2}$$

$$\boxed{= -\frac{2}{3}}$$

<u>Exercise</u> · Compute the example above with a <u>reversed</u> <u>orientation</u>.

· Does the answer make sense?

· Why does orientation matter here even though we can write this as an integral with respect to arc length?

<u>REMARK</u> What is the connection between scalar line integrals and integrals over vector fields?

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, each functions of (x,y,z), $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$= \int_a^b (P\hat{\imath} + Q\hat{\jmath} + R\hat{k}) \cdot (x'(t)\,\hat{\imath} + y'(t)\,\hat{\jmath} + z'(t)\,\hat{k})\,dt$$

$$= \int_a^b P(x(t), y(t), z(t)) \cdot x'(t)\,dt + Q(x(t), y(t), z(t)) \cdot y'(t)\,dt + R(x(t), y(t), z(t)) \cdot z'(t)\,dt$$

$$= \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

* exercise understand each step!