14.6 Directional Derivatives and the gradient vector

REM For $f(x,y)$:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

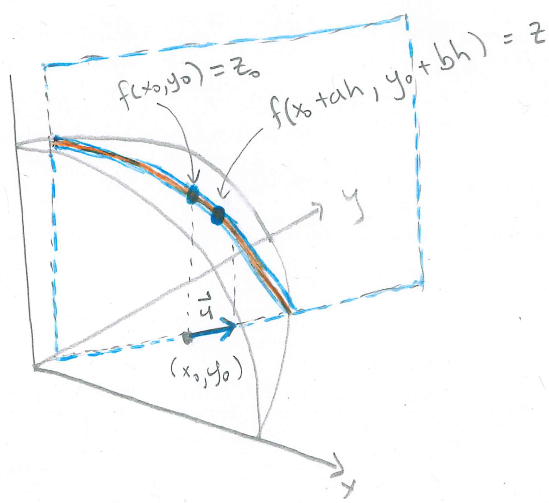
and

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

CLAIM f_x and f_y are derivatives in the direction of $\hat{\imath}$ and $\hat{\jmath}$, respectively. We'll see why we can say this in a bit.

QUESTION What if we wanted to find the rate of change of $z = z(x,y)$ $(= f(x,y))$ at some (x_0, y_0) in an arbitrary direction? As in, not just in the x-direction or y-direction? Let $\vec{u} = \langle a,b \rangle$ be an arbitrary

(*) $\boxed{\text{unit}}$ vector, and let's try to construct it!



Then, $\Delta z = z - z_0 = f(x_0 + ha, y_0 + hb) - f(x_0, y_0)$.

To make this a derivative:

$$\lim_{h \to 0} \frac{\Delta z}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

and we get a derivative <u>in the direction</u> of $\vec{u}$.

In fact, there is a nice geometric interpretation. Provided $\vec{u}$ is a (*) unit vector, this now gives the slope of the tangent line at $f(x_0, y_0)$ along the orange curve cutting through the surface on the last page!

DEF The <u>Directional Derivative</u> of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_u f(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

EXERCISE Explain the claim on the first page of these notes!

So, that's a nice definition ... but if we needed to compute a directional derivative, that would be a hard definition to use. Luckily, if we observe one thing, we can make the computation much simpler:

THM $D_u f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\cdot b$

proof: Define $g(h) = f(x_0 + ha, y_0 + hb)$, where $\vec{u} = \langle a, b\rangle$, a unit vector.

Fix x_0, y_0. Then notice:

① $\lim_{h\to 0} \dfrac{g(h) - g(0)}{h} = g'(0)$

and

② $\lim_{h\to 0} \dfrac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h} = D_u f(x_0,y_0)$.

In other words: $g'(0) = D_u f(x_0,y_0)$.

Next, notice we can compute $g'(h)$ using the chain rule!

$$g(h) = g(x(h), y(h)) \quad \text{where} \begin{cases} x(h) = x_0 + ha \\ y(h) = y_0 + hb. \end{cases}$$

So,

$$\frac{dg}{dh} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dh} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dh}$$

$$g'(h) = f_x(x,y)\cdot a + f_y(x,y)\cdot b$$

Then, $g'(0) = f_x(x_0,y_0)\cdot a + f_y(x_0,y_0)\cdot b$, and if we combine this with ① and ② above:

$$\boxed{D_u f(x_0,y_0) = f_x(x_0,y_0)\cdot a + f_y(x_0,y_0)\cdot b}$$

for every (x_0,y_0).

COR Since $\vec{u}$ is a unit vector, we can write $\vec{u} = \langle \cos\theta, \sin\theta \rangle$ for some θ. Then, the above theorem tells us

$$\boxed{D_u f(x,y) = f_x(x,y)\cdot\cos\theta + f_y(x,y)\cdot\sin\theta}$$

EXAMPLE 1

Find the directional derivative $D_u f(x,y)$ if

$$f(x,y) = x^3 - 3xy + 4y^2$$

and $\vec{u}$ is the unit vector with angle $\pi/6$. What is $D_u f(1,2)$?

STEP 1: $D_u f(x,y) = f_x(x,y)\cdot\cos(\pi/6) + f_y(x,y)\cdot\sin(\pi/6)$

$$= \boxed{(3x^2 - 3y)\cdot\frac{\sqrt{3}}{2} + (-3x+8y)\cdot\frac{1}{2}}$$

STEP 2: $D_u f(1,2) = (3\cdot 1^2 - 3\cdot 2)\frac{\sqrt{3}}{2} + (-3\cdot 1 + 8\cdot 2)\cdot\frac{1}{2}$

$$= \frac{-3\sqrt{3}}{2} + \frac{13}{2} = \boxed{\frac{13-3\sqrt{3}}{2}}$$

$\boxtimes$

Next, observe: $D_u f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\cdot b$

$$= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle$$

$$= \underbrace{\langle f_x(x,y), f_y(x,y)\rangle}\cdot\vec{u}$$

we call this the gradient vector!

__DEF__ Let f be a differentiable function of two variables, $f(x,y)$. The __gradient__ __vector__ is

$$\text{grad } f = \nabla f = \langle f_x, f_y \rangle = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

(*) $\boxed{\text{NOTICE} \quad D_u f(x,y) = \nabla f \cdot \vec{u}}$ (*)

__EXERCISE__ Read Example 3 (14.6) in the book.

__EXAMPLE 2__

Find the directional derivative of $f(x,y) = x^2y^2 - 4y^3$ at the point $(2,-1)$ in the direction of $\vec{v} = 2\hat{\imath} + 5\hat{\jmath}$.

__STEP 1__ $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2xy^2, 2x^2y - 12y^2 \rangle$

$$\vec{v} = \langle 2,5 \rangle \implies \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 2,5 \rangle}{\sqrt{2^2+5^2}} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

(Make it a __unit__ vector!)

__STEP 2__ $D_u f(x,y) = \nabla f \cdot \vec{u}$

$$= \langle 2xy^2, 2x^2y - 12y^2 \rangle \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

$$= \frac{1}{\sqrt{29}}\left(4xy^2 + 10x^2y - 60y^2\right)$$

__STEP 3__ $D_u f(2,-1) = \frac{1}{\sqrt{29}}\left(4(2)(-1)^2 + 10(2^2)(-1) - 60(-1)^2\right)$

$$= \frac{1}{\sqrt{29}}\left(8 + {}^{-}40 + {}^{-}60\right)$$

$$\boxed{= \frac{-92}{\sqrt{29}}}$$

QUESTION What if f is a function of 3 variables?

i.e. $f(x,y,z)$?

Let $\vec{u} = \langle a,b,c \rangle$ be a $\boxed{\text{unit}}$ vector and define:

$$D_u f(x,y,z) := \lim_{h \to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}$$

This will give us (same as before)

$$D_u f(x,y,z) = \nabla f \cdot \vec{u}, \quad \vec{u} \text{ unit vector}$$

where

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

REMARK In fact, we can do this for any number of variables! So this can be defined in n-dimensions!

QUESTION In what direction is the directional derivative maximal?

NOTICE $D_u f(x,y) = \nabla f \cdot \vec{u} = |\nabla f| \cdot |\vec{u}| \cos\theta$

$$= |\nabla f| \cos\theta$$

If $\cos\theta = 1$, we have a max. This occurs when $\theta = 0$. θ here is the angle between between the gradient of f and $\vec{u}$, so the maximal directional derivative occurs when $\vec{u}$ has the same direction as ∇f! The max is $|\nabla f|$!

Let's record this as a Theorem.

<u>THM</u> Let f be a differentiable function of two or three variables (actually n). The maximum value of the directional derivative $D_u f(x_1, x_2, \ldots, x_n)$ is $|\nabla f|$ and it occurs when $\vec{u}$ has the same direction as ∇f.

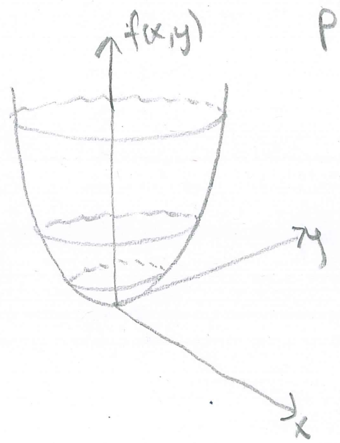
<u>EXERCISE</u> EXAMPLE 7 in the text.

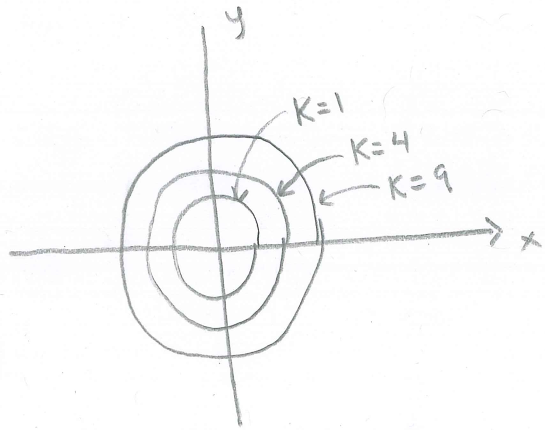
[So, we can describe the gradient of f as the vector that points in the same direction as the <u>direction of fastest increase</u> or the <u>direction of steepest ascent</u>.] (*)

In fact, we can also show that the gradient vector is also a vector perpindicular to level surfaces of $f(x,y,z)$ (or level curves of $f(x,y)$).

DEF Let $f(x,y)$ be a smooth function of two variables. We say a <u>level curve of f</u> is the set of points satisfying $f(x,y) = K$, for some constant K.

EXAMPLE 3 Let $f(x,y) = x^2+y^2$. Then the level curves are points satisfying $x^2+y^2=K$. For $K = 1, 4, 9$, the level curves are plotted below:





(Notice the similarity to traces!)

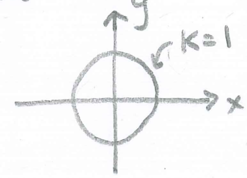
Now, we can parametrize a level curve provided f is a nice enough function, i.e. we want

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

where $f(x(t), y(t)) = K$ for some K.

EXAMPLE 4 Let $f(x,y) = x^2 + y^2$ and pick $K=1$. Then the level curve is a circle of radius 1, and we can parametrize it as follows:

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \end{cases}$$



Then $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle \cos t, \sin t \rangle$

and notice:

$$f(\cos t, \sin t) = \cos^2 t + \sin^2 t = 1$$

for all t.

QUESTION What does it mean for the gradient vector to be orthogonal to a level curve?

At any point on the curve, there is a tangent line, and we want orthogonal to mean the vector is orthogonal to any vector on this line!

QUESTION What vector is on this tangent line?

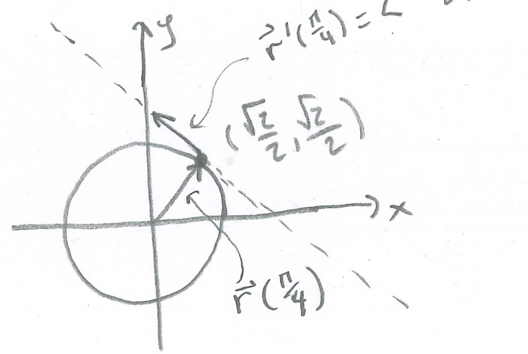
Well, at a point on the level curve, there is some t such that $\vec{r}(t)$ points to that point, and $\vec{r}\,'(t)$ is a vector pointing in the direction of the tangent line!

EXAMPLE 5 Let $f(x,y) = x^2+y^2$, and let $K=1$. Then, let $\vec{r}(t) = \langle \cos t, \sin t \rangle$ be as before. Now, what is a vector that points along the tangent line at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$?

Notice: $\begin{cases} \cos t = \frac{\sqrt{2}}{2} \\ \sin t = \frac{\sqrt{2}}{2} \end{cases} \Rightarrow t = \frac{\pi}{4}$.

Then $\vec{r}\,'(t) = \langle -\sin(t), \cos t \rangle$

and $\vec{r}\,'(\frac{\pi}{4}) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$.



EXAMPLE 6 Notice, $\nabla f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is perpindicular to $\vec{r}\,'(\frac{\pi}{4})$!

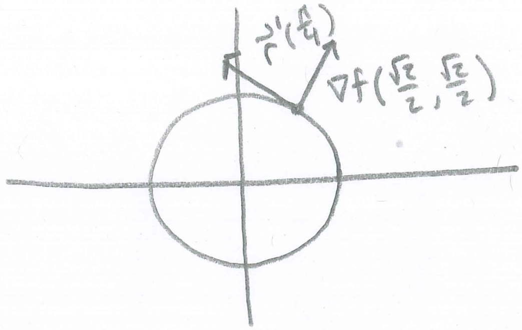
$\nabla f = \langle 2x, 2y \rangle$

$\nabla f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = \langle \sqrt{2}, \sqrt{2} \rangle$

$\vec{r}\,'(\frac{\pi}{4}) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

(!)

and $\langle \sqrt{2}, \sqrt{2} \rangle \cdot \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = -1 + 1 = 0$



QUESTION How would one show this is always the case?

THM The gradient vector is orthogonal to level curves. (Let $f(x,y)$ be a smooth function.)

pf: Fix K for any level curve:

$$f(x,y) = K$$

Parametrize the level curve:

$$r(t) = \langle x(t), y(t) \rangle$$

so

$$f(x,y) = f(x(t), y(t)) = K.$$

Take an implicit derivative with respect to t.
Use the chain rule, and remember K is a constant!

$$\frac{d}{dt}\left(f(x(t), y(t))\right) = \frac{d}{dt}(K)$$

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = 0$$

$$\nabla f \cdot \vec{r}'(t) = 0$$

(!) $\boxed{\nabla f \cdot \vec{r}'(t) = 0}$

The following question is worth 2 test points.

To get the 2 points, you will need to work in a group of at least 2 people, and everyone in the group must understand the answer/solution. You will need to present the solution to the instructor (me!).

Question: Imagine you are in a world in which your instructor mistakenly defines a directional derivative with arbitrary vectors, not just unit vectors. (As if that could ever happen!) The mistake is pointed out, and the instructor retreats to their quarters to fix the lecture notes. Afterwards, being an astute student, you realize the definition still holds meaning.

You define a "pseudo-directional derivative" as a generalization of a directional derivative as follows:

DEF For arbitrary vector $\vec{v}$, the pseudo-directional derivative of a differentiable function $f(x,y)$ is

$$D_{\vec{v}}\, f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\cdot b$$

where $\vec{v} = \langle a,b \rangle$.

(Notice, we could use the limit definition and derive this as a consequence!)

Now, you notice that if the vector $\vec{v}$ is <u>not</u> a unit vector, the pseudo-direction vector does <u>not</u> give you information about the slope of the ~~tangent~~ line ~~to the s~~ tangent to the surface, in the direction of $\vec{v}$. However, you realize you could scale the whole system such that the pseudo-directional derivative you compute gives you the slope of a tangent line at a point on the scaled surface.

What is the point? And how do you scale the surface? You may use the fact that directional derivatives, when defined correctly (i.e. with unit vectors) give you the slope of a tangent line to the surface in the direction of the unit vector.

<u>Hint:</u> How would you make $\vec{v}$ a unit vector?

"Scaling" and "coordinate system" are the key ideas here.