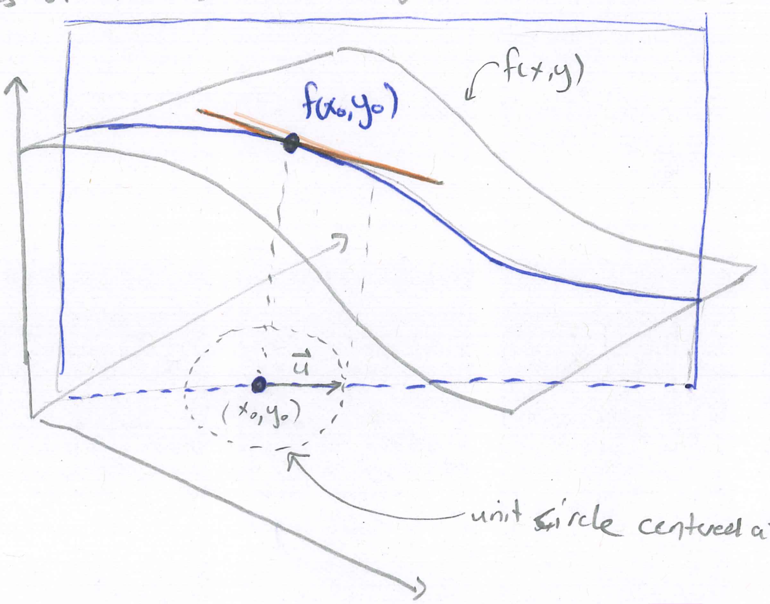
Lecture #10

<u>CORRECTIONS</u>: Last lecture was corrected - see the notes.

We define directional derivatives with <u>unit vectors</u>, since the unit vectors will give us slopes on the surface in question:



$D_{\vec{u}} f(x_0,y_0)$ gives the <u>slope</u> of the tangent line pictured, i.e. the slope of the curve on the surface in the direction of $\vec{u}$.

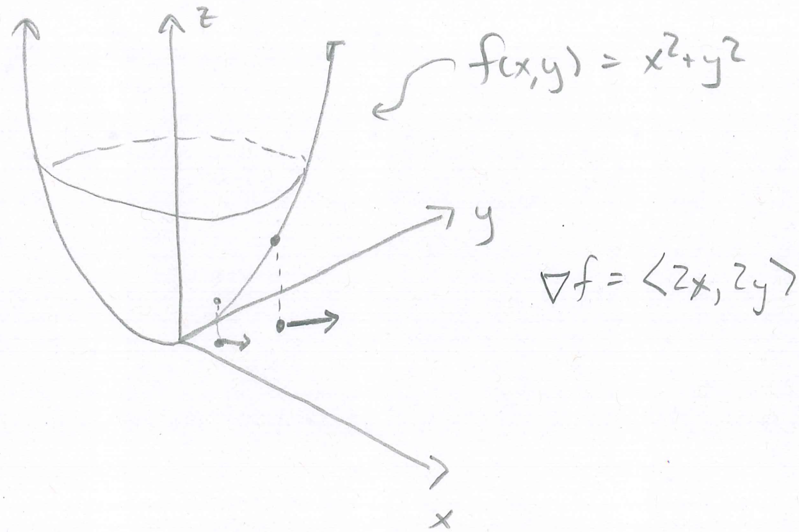
<u>16.1</u> Vector fields

<u>REM</u> $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$, $f(x,y)$.

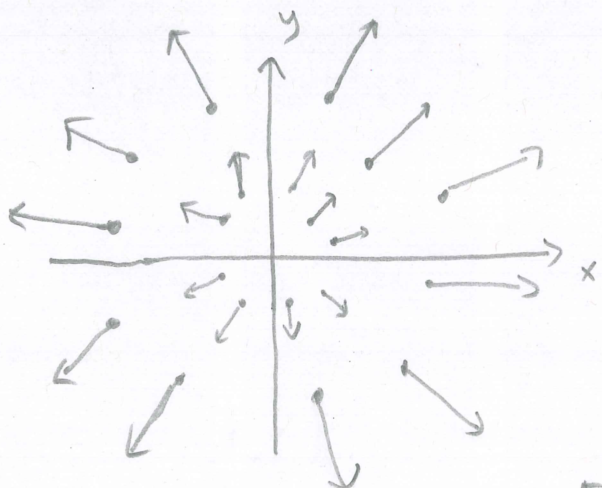
We can use the gradient to assign a vector to each point $(x,y) \in \mathbb{R}^2$. For example, let $f(x,y) = x^2+y^2$.

Then $\nabla f = \langle 2x, 2y \rangle$, so at each (x,y), we could attach a vector $\langle 2x, 2y \rangle$.

PIC



In the xy-plane,



Notice: The gradient "vector field" always points in the direction that the $f(x,y)$ increases the fastest!

DEF Let D be a set in $\mathbb{R}^2$ (a plane region). A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x,y) in D a two-dimensional vector $\vec{F}(x,y)$.

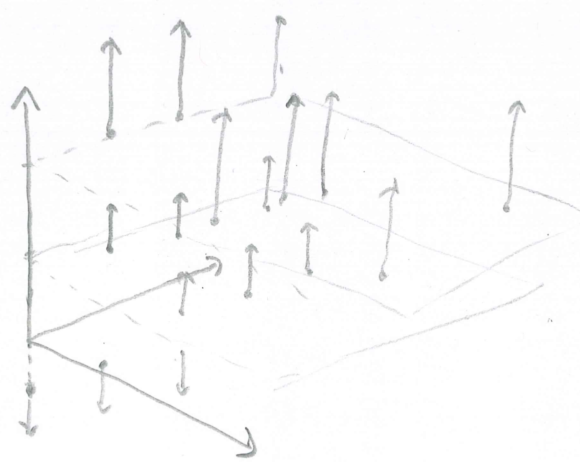
DEF Let E be a subset of $\mathbb{R}^3$. A <u>vector field</u> <u>on $\mathbb{R}^3$</u> is a function $\vec{F}$ that assigns to each point (x,y,z) in E a three dimensional vector $\vec{F}(x,y,z)$.

REMARK We often write $\vec{F}(x,y)$ as $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$, and call $P(x,y)$ and $Q(x,y)$ <u>component functions</u>. Notice that $P(x,y)$ and $Q(x,y)$ are <u>scalar</u> <u>functions</u>. We sometimes call these <u>scalar</u> <u>fields</u>. (Assign a scalar to each point instead of a vector!)

EXAMPLE 1 Sketch the vector field on $\mathbb{R}^3$ given by

$$\vec{F}(x,y,z) = z\,\hat{k}$$

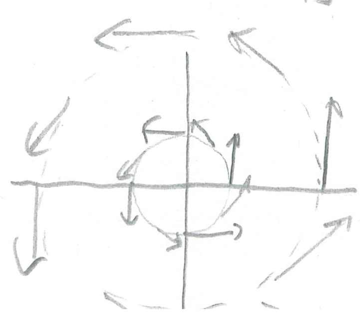
eg.



EXAMPLE 2 Sketch the vector field on $\mathbb{R}^2$ defined by

$$\vec{F}(x,y) = -y\,\hat{\imath} + x\,\hat{\jmath}$$

eg.



(Make a table if it makes it easier for you!)

Exercise Do you recognize this vector field as something you've seen before? Why are there circles drawn on this vector field?

EXAMPLE 3 (Gravitational Field)

rem
$$|F| = \frac{m \cdot M G}{r^2}$$

masses of two objects

G ← gravitational constant

r is the distance between the two objects.

$|F|$: gravitational force

We can write the vector version! Let M be the mass of an object at the origin. Then, the force experienced by an object of mass m at a point (x,y,z) is

$$\vec{F}(\vec{x}) = \frac{mMG}{|\vec{x}|^3} \cdot \vec{x} \quad , \text{ where } \vec{x} = \langle x,y,z \rangle.$$

(Notice, $r = |\vec{x}|$, so $r^2 = |\vec{x}|^2$. The gravitation force exerted on the object at $\vec{x}$ is in the direction of the origin. A unit vector in that direction is given by $-\frac{\vec{x}}{|\vec{x}|}$.)

Breaking this into components: $\left(|\vec{x}| = \sqrt{x^2+y^2+z^2} \right)$

$$\vec{F}(x,y,z) = \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}\,\hat{\imath} + \frac{-mMG\,y}{(x^2+y^2+z^2)^{3/2}}\,\hat{\jmath} + \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}}\,\hat{k}$$

EXERCISE Show this! (compute).

Then, this gives us a vector field representative of a gravitational field.

EXERCISE Work through EXAMPLES in 16.1. (Electric Fields!)

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DEF We say a vector field $\vec{F}$ is a conservative vector field if it is the gradient of some scaler function, i.e. there exists some $f(x,y)$ such that $\nabla f = \vec{F}$.

REMARK Clearly then, any "gradient vector field" is a conservative vector field.

EXERCISE Of the examples in the notes, which are conservative vector fields? Which are not? How can you tell?

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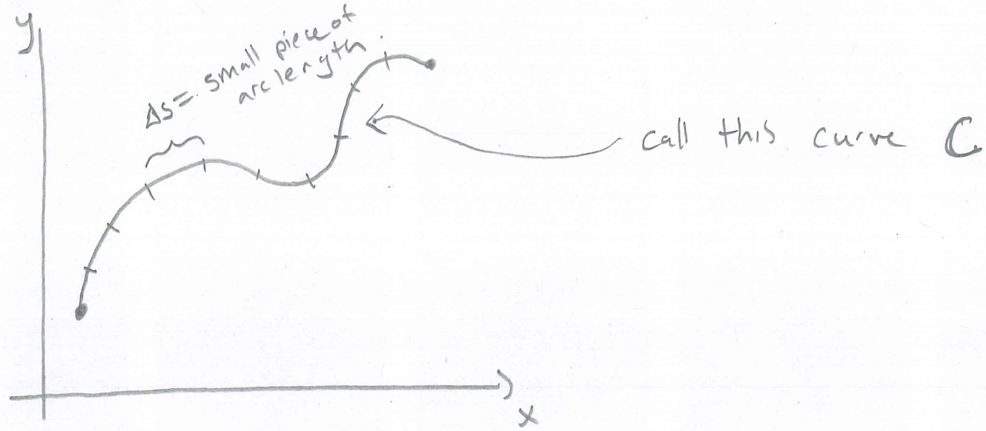
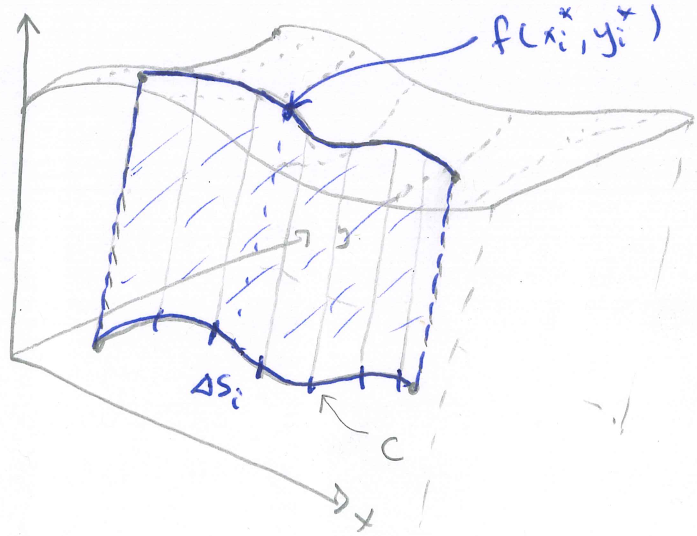
Our next goal is to learn to compute line integrals, specifically line integrals through vector fields (think of the vector field as doing work on a particle as it follows some trajectory in the vector field). First, however, we need to develop some notions.

16.2 Line Integrals

Let's start with a scalar function version. Say we have some nice function $f(x,y)$, and a curve in the (x,y) plane:



At each point (x,y) along this curve, $f(x,y)$ is some number, so we have:



If we wanted to compute the area of the "curtain" in **BLUE!** above, then we would need to compute

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \cdot \underline{\Delta s_i}$$

small piece of arc length

height associated with the "small piece of" arc length.

The limit of this becomes an integral:

$$\boxed{\int_C f(x,y)\,ds}$$

where C is the curve / trajcetory and ds an infinitesimally small piece of arc length.

<u>QUESTION</u> How do you compute such a thing??

First, recall arc length. (Even better ... surface area!)

$$SA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \quad \text{(from 15.5)}$$

$$\text{arc length} = L = \int_C \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{(from Ch. 10.2)}$$

<u>If</u> C is a parametrized curve, in other words

$$\begin{cases} x(t) \\ y(t) \end{cases}$$

Then, notice (NOT RIGOROUS!)

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \left(\sqrt{\frac{dx^2 + dy^2}{dx^2}}\right) \cdot dx = \sqrt{dx^2 + dy^2}$$

and,
$$\begin{cases} x(t) \\ y(t) \end{cases} \rightsquigarrow \quad dx = \frac{dx}{dt} \cdot dt$$
$$dy = \frac{dy}{dt} \cdot dt \; ,$$

so
$$\sqrt{dx^2 + dy^2} = \sqrt{dt^2\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(NOT RIGOROUS! More like a mnemonic device! See 10.2 in the text for more explanation if you'd like!)

Then,
$$\boxed{\int_C f(x,y)\, ds = \int_{t=a}^{t=b} f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

So to begin, most of the time, our first challenge in computing line integrals is representing the curve C as a parametric equation in t!

EXAMPLE 1   Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle.

<u>STEP 1</u> Parametrize C!

Here, we have a circle:

$$\left.\begin{aligned} x(t) &= r\cos t \\ y(t) &= r\sin t \end{aligned}\right\} \quad r=1 \text{ since this is a unit circle.}$$

Notice, $(x(0), y(0)) = (1,0)$ and as t increases, $(x(t), y(t))$ moves <u>counterclockwise</u>.

$\Rightarrow$ If you want a refresher on parametrizing curves, come talk to me! For circles/ellipses, you need to remember the formula above. You need to know how to start at an arbitrary position on a circle, and move either clockwise or counterclockwise!

So we get $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$ from <u>$t=0$ to $t=\pi$</u>

<u>STEP 2</u> Plug and chug:

$$\int_C (2 + x^2 y)\, ds = \int_{t=0}^{t=\pi} (2 + \cos^2 t \cdot \sin t) \cdot \left(\sqrt{\overbrace{\sin^2 t}^{\left(\frac{dx}{dt}\right)^2} + \overbrace{\cos^2 t}^{\left(\frac{dy}{dt}\right)^2}} \right) dt$$

$$= \int_0^{2\pi} (2 + \underbrace{\cos^2 t \cdot \sin t}_{\text{u-sub, careful! you get a negative!}})\, dt \qquad \begin{aligned} u &= \cos t \\ du &= -\sin t\, dt \end{aligned}$$

$$= \left[2t + \frac{-\cos^3 t}{3} \right]_0^{\pi}$$

$$= \left(2\pi + \tfrac{1}{3}\right) - \left(0 - \tfrac{1}{3}\right) = \boxed{2\pi + \frac{2}{3}}$$