Lecture #9

14.6 Directional Derivatives and the gradient vector

REM For $f(x,y)$:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,\, y_0) - f(x_0,y_0)}{h}$$

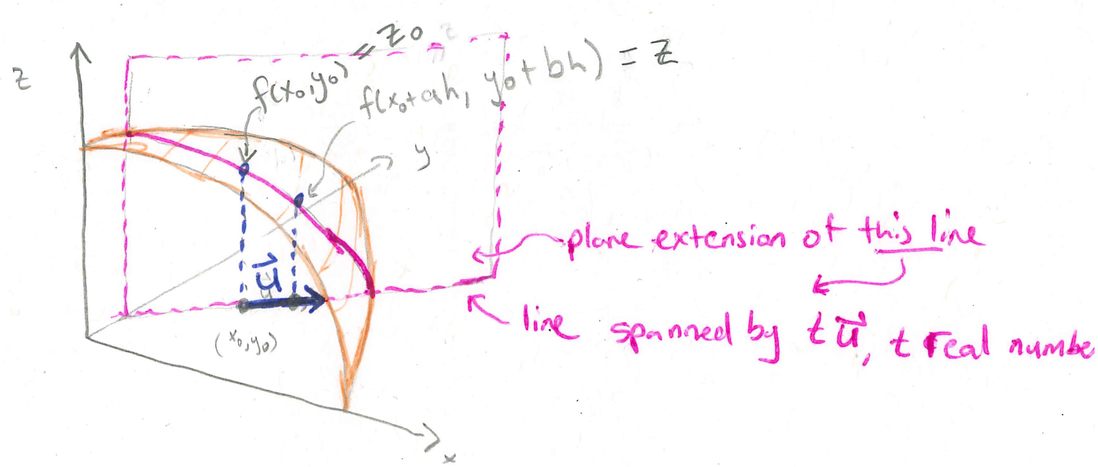
and

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,\, y_0+h) - f(x_0,y_0)}{h}$$

CLAIM f_x and f_y are derivatives in the direction of $\hat{\imath}$ and $\hat{\jmath}$, respectively. We'll see why we can say this in a bit.

QUESTION What if we wanted to find the rate of change of $z = z(x,y)$ $(= f(x,y))$ at some (x_0,y_0) in the direction of an arbitrary unit vector? Let $\vec{u} = \langle a,b \rangle$ be the arbitrary unit vector.

Let's try to construct it:



Then, $\Delta z = z - z_0 = f(x_0+ha, y_0+hb) - f(x_0,y_0)$.

To make this a derivative:

$$\lim_{h\to 0} \frac{\Delta z}{h} = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

And we get a derivative <u>in</u> <u>the</u> <u>direction</u> of $\vec{u}$.

<u>DEF</u> Directional Derivative of f at (x_0,y_0) in the direction of $\vec{u} = \langle a,b\rangle$, where $\vec{u}$ is a <u>unit vector</u>, is

$$D_u f(x_0,y_0) := \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

* This now gives the slope of the tangent line at $f(x_0,y_0)$ along the <u>pink</u> curve cutting through the surface on the last page!

<u>Exercise</u> Explain the claim on the first page of these notes!

So, that's a nice definition ... but if we needed to compute a directional derivative, that would be a hard definition to use.

Luckily, if we observe one thing, we can make the computation much simpler:

<u>THM</u> $D_u f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\cdot b$

proof: Define $g(h) = f(x_0 + ha, y_0 + hb)$, where $\vec{u} = \langle a, b \rangle$ (unit vector!)
Fix x_0, y_0.

Notice: $\lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0)$

and $\lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = D_u f(x_0, y_0)$

in other words: $g'(0) = D_u f(x_0, y_0)$.

Next, notice we can compute $g'(h)$ using the chain rule!

$$g(h) \overset{\text{"}}{=} g(x,y) \overset{\text{"}}{=} f(x,y) \quad \text{where} \quad \begin{array}{l} x(h) = x_0 + ha \\ y(h) = y_0 + hb \end{array} \quad (!)$$

so $$\frac{dg}{dh} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dh} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dh}$$

$$g'(h) = f_x(x,y) \cdot a + f_y(x,y) \cdot b.$$

Then $g'(0) = f_x(x_0, y_0) \cdot a + f_y(x_0, y_0) \cdot b$, which, combining this with what we discovered above, we see

$$\boxed{D_u f(x_0, y_0) = f_x(x_0, y_0) \cdot a + f_y(x_0, y_0) \cdot b}$$

for every (x_0, y_0).

COR ~~If~~ Since $\vec{u}$ is a unit vector, we can write $\vec{u} = \langle \cos\theta, \sin\theta \rangle$ for some θ. Then, the above theorem tells us

$$(*) \quad \boxed{D_u f(x,y) = f_x(x,y) \cdot \cos\theta + f_y(x,y) \cdot \sin\theta} \quad (*)$$

EXAMPLE 1

Find the directional derivative $D_u f(x,y)$ if

$$f(x,y) = x^3 - 3xy + 4y^2,$$

and $\vec{u}$ is the unit vector with angle $\pi/6$. $D_u f(1,2) = ?$

STEP 1: $D_u f(x,y) = f_x(x,y) \cdot \cos(\pi/6) + f_y(x,y) \cdot \sin(\pi/6)$

$$= (3x^2 - 3y) \cdot \left(\frac{\sqrt{3}}{2}\right) + (-3x + 8y) \cdot \left(\frac{1}{2}\right)$$

STEP 2: $D_u f(1,2) = (3 \cdot 1^2 - 3 \cdot 2) \cdot \frac{\sqrt{3}}{2} + (-3 \cdot 1 + 8 \cdot 2) \cdot \frac{1}{2}$

$$= \frac{-3\sqrt{3}}{2} + \frac{13}{2} = \frac{13 - 3\sqrt{3}}{2}$$

Next, notice: $D_u f(x,y) = f_x(x,y) \cdot a + f_y(x,y) \cdot b$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle$$

$$= \underbrace{\langle f_x(x,y), f_y(x,y) \rangle}_{} \cdot \vec{u}$$

we call this the gradient vector!

DEF Let f be a ~~smooth~~ (differentiable) function of two variables, $f(x,y)$. The gradient vector is

$$\text{grad } f = \nabla f = \langle f_x, f_y \rangle = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

Exercise Read Example 3 (14.6) in the book.

EXAMPLE 3 Find the directional derivative of $f(x,y) = x^2y^2 - 4y^3$ at the point $(2,-1)$ in the direction of $\vec{v} = 2\hat{\imath} + 5\hat{\jmath}$.

STEP 1 $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2xy^2, 2x^2y - 12y^2 \rangle$

$\vec{v} = \langle 2,5 \rangle \Rightarrow \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 2,5 \rangle}{\sqrt{29}} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$

NEED THIS TO BE A UNIT VECTOR!

STEP 2 $D_u f(x,y) = \left(\langle 2xy^2, 2x^2y - 12y^2 \rangle \cdot \langle 2,5 \rangle\right) \frac{1}{\sqrt{29}}$

$= \left(4xy^2 + 10x^2y - 60y^2\right) \frac{1}{\sqrt{29}}$

could plug in $(x,y) = (2,-1)$ here.

STEP 3 $D_u f(2,-1) = \left(4(2)(-1)^2 + 10(2)^2 \cdot (-1) - 60(-1)^2\right) \frac{1}{\sqrt{29}}$

$= (8 - 40 - 60) \frac{1}{\sqrt{29}}$

$\boxed{= (-92)} \cdot \frac{1}{\sqrt{29}} = \frac{-92}{\sqrt{29}}$

QUESTION What if f is a function of 3-variables?

i.e. $f(x,y,z)$?

also need $\vec{u}$ to be a unit vector.

Let $\vec{u} = \langle a,b,c \rangle$ and define:

$$D_u f(x_0,y_0,z_0) := \lim_{h \to 0} \frac{f(x_0+ha,\, y_0+hb,\, z_0+hc) - f(x_0,y_0,z_0)}{h}$$

This will give us (same as before)

$$D_u f(x,y) = \nabla f \cdot \vec{u}$$, $\vec{u}$ a unit vector!

where

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

REMARK In fact, we can do this for any number of variables! So this can become n-dimensional!

QUESTION In what direction is the directional derivative maximal?

$$D_u f(x,y) = \nabla f \cdot \vec{u} = |\nabla f| \cdot |\vec{u}| \cos\theta$$

$$= |\nabla f| \cdot 1 \cdot \cos\theta$$

If $\cos\theta = 1$, we have a max. This occurs when $\theta = 0$ or 2π, but this is the same angle (between the gradient of f and $\vec{u}$), so the maximal directional derivative occurs when $\vec{u}$ has the same direction as ∇f ! The max is $|\nabla f|$!

$\Rightarrow$

THM Let f be a differentiable function of two or three variables (actually n). The maximum value of the directional derivative $D_u f(x,y,..)$ is $|\nabla f|$ and it occurs when $\vec{u}$ has the same direction as ∇f.

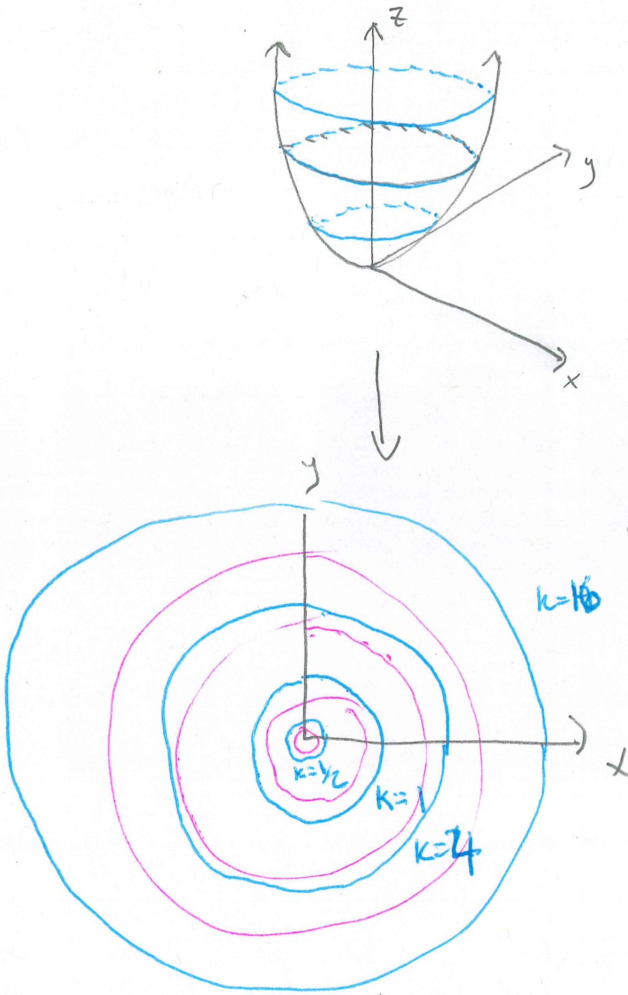
EXERCISE Example 7 in the text.

So, we can describe the gradient of f as the vector that points in the same direction as the direction of fastest increase.

In fact, we could also show (though we won't), that the gradient vector is also a vector perpindicular to the level surfaces of $f(x,y,z)$ (or level curves of $f(x,y)$). Consider the following example.

Let $f(x,y) = x^2 + y^2$. Level curves of this surface are curves where $f(x,y)$ stays constant. Let

$$f(x,y) = k = \underline{x^2 + y^2}$$



these are level curves!

Circles of radius $\underline{\underline{\sqrt{k}}}$!

<u>Exercise</u>: Choose a few points on a level curve. Plot the gradient vector on the xy-plane above. Is it orthogonal to the level curve?

The following question is worth <u>2 test points</u>.

To get the 2 points, you will need to work in a group of at least 2 people, and everyone in the group must understand the answer/solution. You will need to present the solution to the instructor (me!).

<u>Question</u>: Imagine you are in a world in which your instructor mistakenly defines a directional derivative with arbitrary vectors, not just unit vectors. (As if that could ever happen!) The mistake is pointed out, and the instructor retreats to their quarters to fix the lecture notes. Afterwards, being an astute student, you realize the definition still holds meaning.

You define a "pseudo-directional derivative" as a generalization of a directional derivative as follows:

<u>DEF</u> For arbitrary vector $\vec{v}$, the <u>pseudo-directional derivative</u> of a differentiable function $f(x,y)$ is

$$D_{\vec{v}} f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\cdot b$$

where $\vec{v} = \langle a, b\rangle$.

(Notice, we could use the limit definition and derive this as a consequence!)

Now, you notice that if the vector $\vec{v}$ is not a unit vector, the pseudo-direction vector does not give you information about the slope of the ~~tangent~~ line ~~to the s~~ tangent to the surface, in the direction of $\vec{v}$. However, you realize you could scale the whole system such that the pseudo-directional derivative you compute gives you the slope of a tangent line at a point on the scaled surface.

What is the point? And how do you scale the surface? You may use the fact that directional derivatives, when defined correctly (i.e. with unit vectors) give you the slope of a tangent line to the surface in the direction of the unit vector.

<u>Hint:</u> How would you make $\vec{v}$ a unit vector?

"Scaling" and "coordinate system" are the key ideas here.