Lecture #8

15.9 Change of variables in Multiple Integrals

REM U-sub

$$\int_0^{\pi} \sin(x^2)\cdot 2x\,dx = \int_{u(0)=0}^{u(\pi)=\pi^2} \sin(u)\,du$$

u-sub: $u = x^2$
$du = 2x\,dx$

Notice that since the x-bounds are positive, we can write

$$x = \sqrt{u} = x(u)$$

$$dx = \frac{1}{2}u^{-1/2}\,du = x'(u)\,du$$

Then

$$\int_0^{\pi} \sin(x^2)\cdot 2x\,dx = \int_{\boxed{x^{-1}(0)=0}}^{\boxed{x^{-1}(\pi)=\pi^2}} \sin((\overset{x(u)}{\sqrt{u}})^2)\cdot 2\overset{x(u)}{\sqrt{u}}\cdot \underbrace{\frac{1}{2}u^{-1/2}\,du}_{x'(u)\,du}$$

$$= \int_0^{\pi} \sin(u)\,du.$$

NOTICE We can think of u-sub as a change of coordinates from x-coordinates to u-coordinates. Here, we think of expressing this change by pick some $x(u)$ (a function!).

i.e.

$$\boxed{\int_a^b f(x)\,dx = \int_{x^{-1}(a)=c}^{x^{-1}(b)=d} f(x(u))\cdot x'(u)\,du}$$

<u>REMARK</u> This equation, which comes from the idea of u-substitution, gives us a way to change from x-coordinates to u-coordinates in the one-dimensional case.

Our goal is to extend this idea into n-dimensions! We'll start with 2 though...

<u>REM</u> Polar Coordinates! (We have already done this, we just hadn't realized it.

$$\begin{cases} x(r,\theta) = r\cos\theta \\ y(r,\theta) = r\sin\theta \end{cases}$$

And integration became:

$$\iint_R f(x,y)\,dx\,dy = \iint_{\boxed{S}_{①}} f(r\cos\theta, r\sin\theta)\,\overset{②}{\boxed{r}}\,dr\,d\theta$$

① S is R written in polar coordinates

② This has something to do with a derivative!

We will focus on ① for a moment, and hold off on the derivative issue (②). If we change coordinates to something other than polar, we need to know how to rewrite the region R in new coordinates.

EXAMPLE 1

Consider the coordinates defined by

$$x = u^2 - v^2$$
$$y = 2uv$$

Find the image of the square $S = \{(u,v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$.

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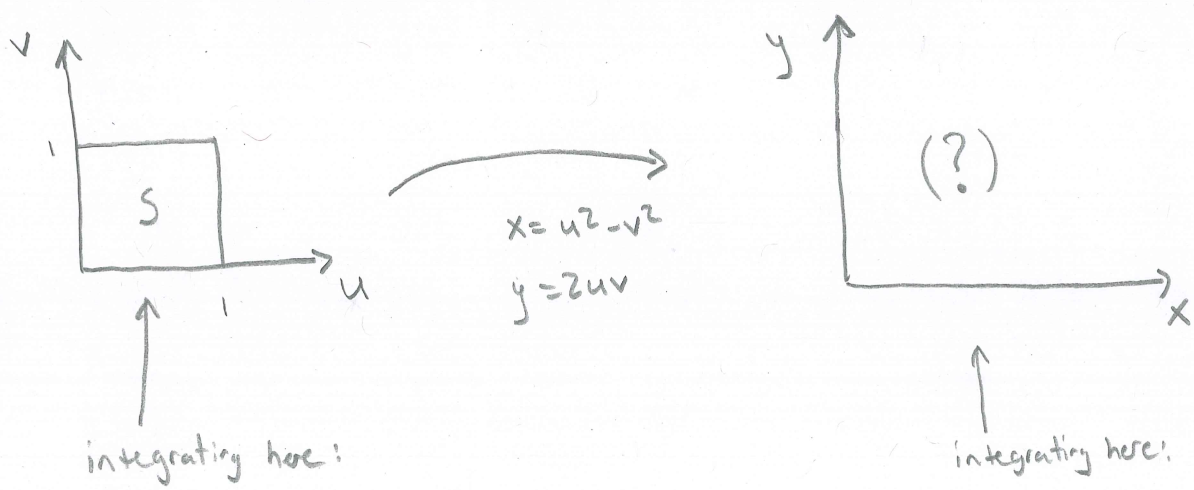
NOTICE We are going backwards! S is given in the new coordinates (u,v), i.e. S is given in the u-v plane!



$$\iint_S f(x(u,v), y(u,v)) \cdot \begin{bmatrix}\text{some derivative} \\ \text{thing}\end{bmatrix} \cdot du\,dv = \iint_{(?)} f(x,y)\,dx\,dy$$

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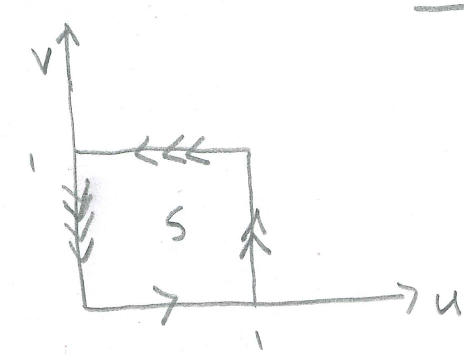
To find the image of S under $x = u^2 - v^2$ and $y = 2uv$, we need to figure out where the edges of S go, so let's label them:

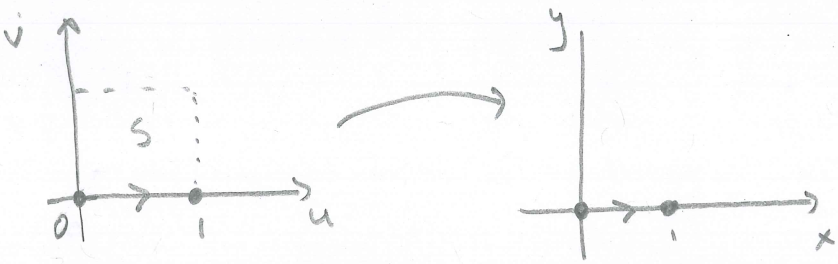


Start with $\rightarrow$, where $v=0$, $0 \le u \le 1$. Plug it into our equation:

$$x = u^2 - \overset{(v^2)}{0} = u^2$$

$$y = 0$$

So, along this first edge it gets mapped to $x = u^2$ for u being from 0 to 1, which means x goes from 0 to 1. Notice, y is zero the whole time!



Next, lets do $\uparrow$, where $u=1$ and $0 \le v \le 1$. Plug it into our equations:

$$x = u^2 - v^2 = 1 - v^2$$

$$y = 2v$$

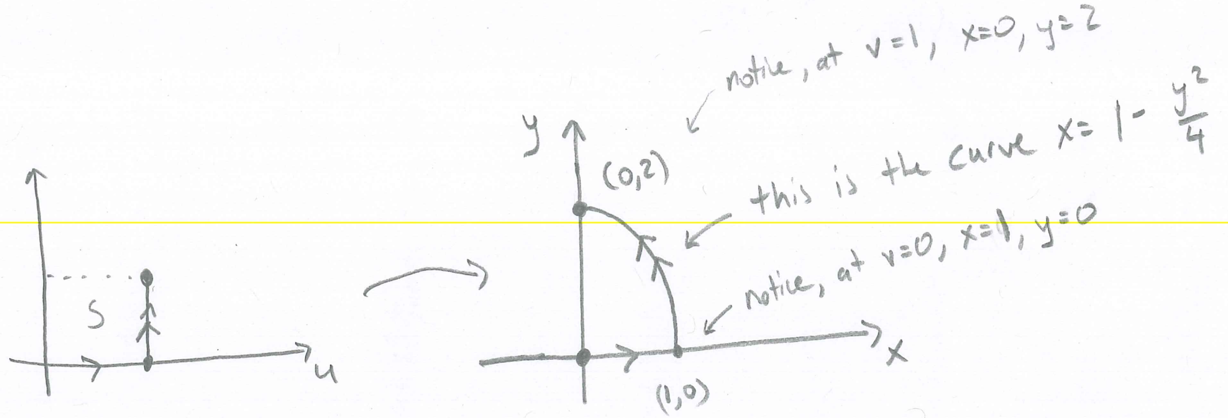
Notice that both x and y depend only on v, so we can treat this like a system of equations! Notice $y = 2v \Leftrightarrow v = \frac{1}{2}y$.

Then

$$x = 1 - v^2 = 1 - \left(\tfrac{1}{2}y\right)^2 = 1 - \frac{y^2}{4}.$$

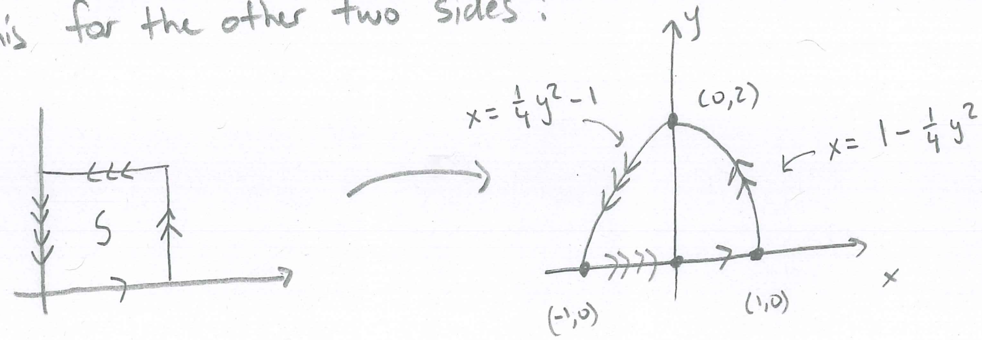
$$\left(\text{or } y^2 = 4(1-x) \Leftrightarrow y = 2\sqrt{1-x}\right)$$

Then,



EXERCISE Do this for the other two sides!

solution:

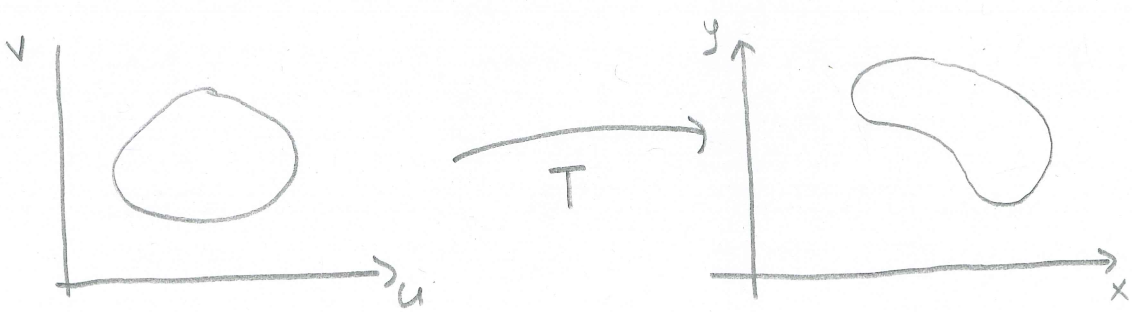


REMARK We typically think of

$$x = x(u,v)$$
$$y = y(u,v)$$

as a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, i.e.

$T(u,v) = (x(u,v), y(u,v))$, where



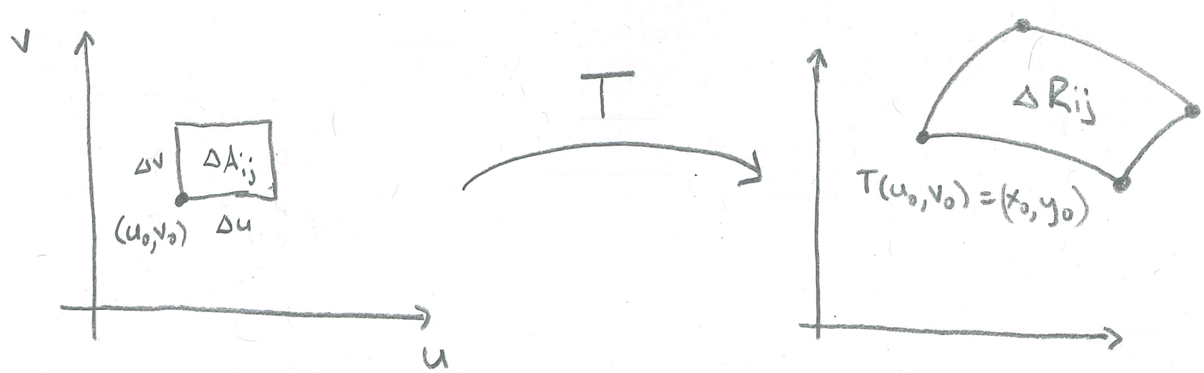
T maps from the <u>uv-plane</u> to the <u>xy-plane</u>.

(*)

For this to work, we need T to be a C^1-transformation, in other words, it needs to have all partial derivaties of order 1, and the map must be <u>one-to-one</u> onto the image. This property means there is an inverse transformation, which we want! (Why?) More to the point, we really want the map one-to-one... otherwise two "coordinates" in the uv-plane get sent to the same "coordinate" in the xy-plane.

Now, using this idea of a transformation, we need to figure out ② ... how the derivative plays a role.

To understand this, we need to understand how a small piece of area $\Delta A = \Delta u \cdot \Delta v$ in the uv-plane produces a small piece of area in the xy-plane under a transformation (or change of coordinates).



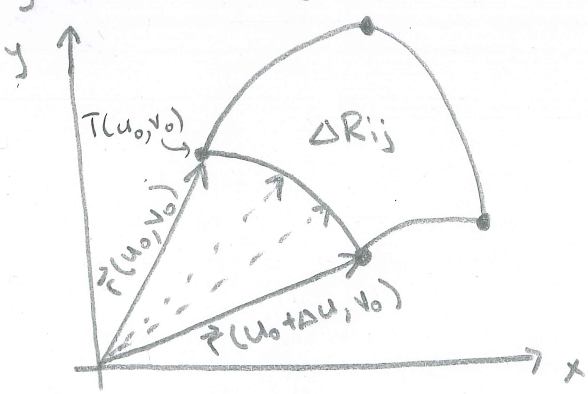
Now, look closely at ΔR_{ij}. Can we define a function along the boundary? Yes! By defining a <u>vector-valued function</u> on the region.

$$T(u,v) = (x(u,v), y(u,v))$$

so define

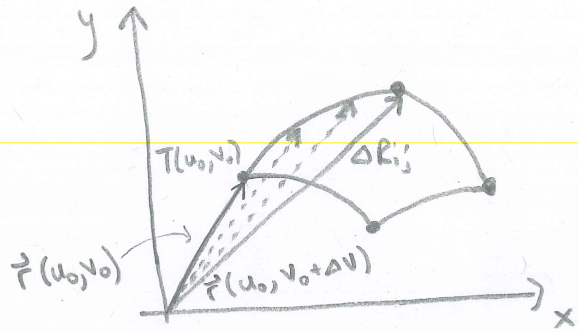
$$\vec{r}(u,v) = \langle x(u,v), y(u,v) \rangle$$
$$= x(u,v)\, i + y(u,v)\, j.$$

Notice, if we fix $v = v_0$, and let u go between u_0 and $u_0 + \Delta u$, we see $\vec{r}(u,v)$ is a vector that moves along one of the edges of ΔR_{ij}:



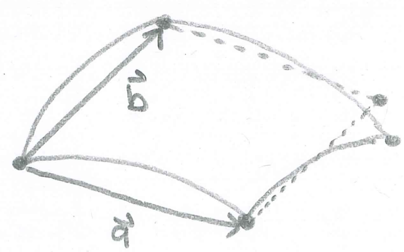
($\vec{r}(u_0, v_0)$ points to $T(u_0, v_0)$)

Similarly, if we fix $u = u_0$ and let v go between v_0 and $v_0 + \Delta v$, we see $\vec{r}(u,v)$ is a vector that moves along another edge of ΔR_{ij}:



<u>EXERCISE</u> If the sides of ΔR_{ij} mapped above are correct for the given transformation, can you describe the other two sides?

Now, notice that as ΔR_{ij} gets small (when $\Delta u \cdot \Delta v$ gets small), we can approximate the area of ΔR_{ij} by computing the area of a parallelogram:



i.e., Area $(\Delta R_{ij}) \approx |\vec{a} \times \vec{b}|$. Notice that we know what $\vec{a}$ and $\vec{b}$ are! Use the images above and properties of vector addition to show:

$$\boxed{\begin{aligned} \vec{a} &= \vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0) \\ \vec{b} &= \vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0) \end{aligned}}$$

Now, notice that as $\Delta u \to 0$ and $\Delta v \to 0$, we can actually give a "limiting" characterization of $\vec{a}$ and $\vec{b}$.

Consider

$$\lim_{\Delta u \to 0} \frac{\vec{a}}{\Delta u} = \frac{\vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0)}{\Delta u} = \vec{r}_u$$

partial derivative with respect to u!

$$\lim_{\Delta v \to 0} \frac{\vec{b}}{\Delta v} = \frac{\vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0,v_0)}{\Delta v} = \vec{r}_v$$

partial derivative with respect to v!

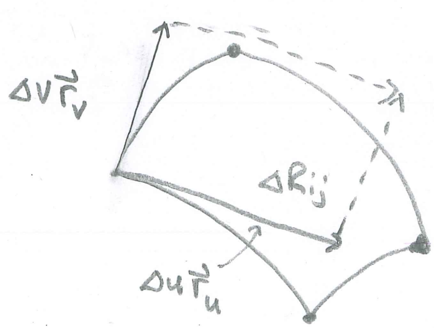
So, as Δu and Δv get small,

$$\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0) \approx \boxed{\Delta u\, \vec{r}_u}$$

and

$$b = \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0,v_0) \approx \boxed{\Delta v\, \vec{r}_v}$$

i.e.



and for small $\Delta u, \Delta v$

$$\boxed{\text{Area}(\Delta R_{ij}) \approx |\Delta u\, \vec{r}_u \times \Delta_v \vec{r}_v|}$$

So, to approximate the area of ΔR_{ij}, we want to compute

$$\left| \Delta u \, \vec{r}_u \times \Delta v \, \vec{r}_v \right| .$$

First, notice Δu and Δv are just numbers! i.e. these are scalar quantities, so

$$\left| \Delta u \, \vec{r}_u \times \Delta v \, \vec{r}_v \right| = |\vec{r}_u \times \vec{r}_v| \cdot \Delta u \Delta v .$$

Then, recall that since

$$\vec{r}(u,v) = \langle x(u,v), y(u,v) \rangle$$

we have

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle = \frac{\partial x}{\partial u} \hat{\imath} + \frac{\partial y}{\partial u} \hat{\jmath}$$

and

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle = \frac{\partial x}{\partial v} \hat{\imath} + \frac{\partial y}{\partial v} \hat{\jmath} .$$

Thus, we can compute:

$$|\vec{r}_u \times \vec{r}_v| = \left\| \begin{matrix} \hat{\imath} & \hat{\jmath} & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{matrix} \right\|$$

Double lines!
(Want the <u>magnitude</u> of the cross product)

$$= \left| \left(0\cdot\frac{\partial y}{\partial u} - 0\cdot\frac{\partial y}{\partial v}\right)\hat{\imath} + \left(0\cdot\frac{\partial x}{\partial v} - 0\cdot\frac{\partial x}{\partial u}\right)\hat{\jmath}\right.$$

$$\left. + \left(\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial v}\right)\hat{k}\,\right|$$

(*) magnitude of cross product!

notice! This is the determinant of a matrix!

$$= \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{k} \right| = \left| \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} \right|$$

This is actually a very important matrix, so we will give it a name.

DEF We say the Jacobian of a transformation is

$$J = \frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}.$$

REMARK Typically, one calls the 2x2 matrix the Jacobian, and the quantity above the determinant of the Jacobian, but we will stick to Stewart's terminology.

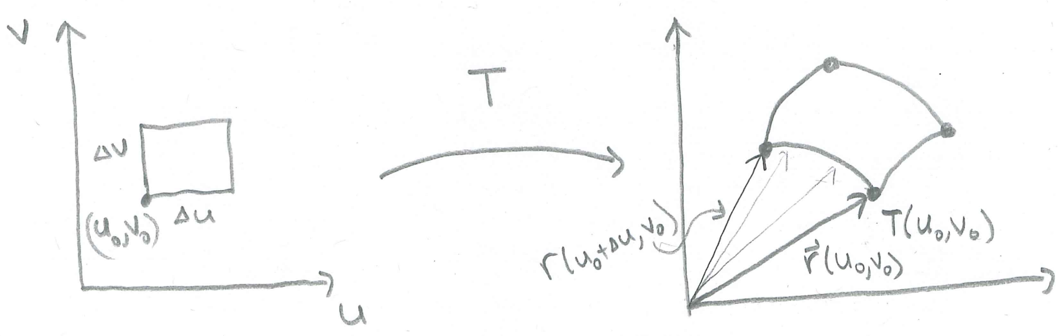
Now, we can say a bit more. A small piece of area in the xy-plane looks like

$$\Delta A = \text{Area}(\Delta R_{ij}) \approx |J| \cdot \Delta u \Delta v$$
$$= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

where here $|\cdot|$ denotes the absolute value. In the limit, this becomes

$$(*) \quad \boxed{dx\,dy = dA = |J|\,du\,dv = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv} \quad (*)$$

REMARK Why do we need to take the absolute value? This is because our transformation may have reversed the orientation, for example:



(u "goes backwards" now!)

This will cause $\vec{a} \times \vec{b}$ above to be negative, This is why we considered the magnitude (plus, that's the area of the parallelogram!).

Now, we can finally say what we mean by change of variables in higher dimensions. The Jacobian is how we solve ② above ... the derivative question.

DEF Change of Variable in a Double Integral

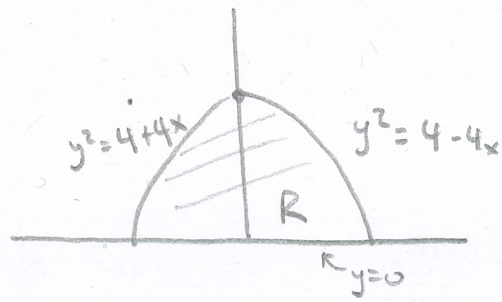
$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv.}$$

NOTE We need the Jacobian to be non-zero, i.e. $T(u,v)=(x(u,v),y(u,v))$ must be an invertible transformation!

EXAMPLE 2 Use the change of variables $x=u^2-v^2$, $y=2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2=4-4x$ and $y^2=4+4x$, for $y\geq 0$.

<u>STEP 1</u>: Notice! There is a strong relationship to Example 1.

For $T(u,v) = (x(u,v), y(u,v))$ as above, and the region R:



We know $T^{-1}(R)$! It's the square S!

$$S = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}.$$

<u>STEP 2</u>: Compute the Jacobian! And take the absolute value!

outer lines are the absolute value

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| = \left|\begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}\right|$$

$$= |4u^2 + 4v^2|$$

$$= 4u^2 + 4v^2$$

(inside quantity is always positive!)

STEP 3: Compute the integral !

$$\iint_R y\,dA = \iint_S \underbrace{(2uv)}_{y}\cdot|J|\,du\,dv = \int_0^1\int_0^1 2uv(4u^2+4v^2)\,du\,dv$$

$$= 8\int_0^1\int_0^1 (u^3v+uv^3)\,du\,dv$$

$$= 8\int_0^1 \left[\tfrac{1}{4}uv + \tfrac{1}{2}u^2v^3\right]_0^1 dv$$

$$= 8\int_0^1 \left(\tfrac{1}{4}v + \tfrac{1}{2}v^3\right) dv$$

$$= 8\left(\left[\tfrac{1}{8}v^2 + \tfrac{1}{8}v^4\right]_0^1\right)$$

$$\boxed{= 2}$$