Lecture #8

14.5 Chain Rule

REM Single variable case:

$$f(x) = g(h(x))$$

$$\frac{df}{dx} = \frac{dg}{d(h(x))} \cdot \frac{dh}{dx}$$

or $$\boxed{f'(x) = g'(h(x)) \cdot h'(x)}$$

We can write this as follows. Let $y = f(x)$ and $x = g(t)$. Then, provided these are differentiable functions,

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

$$\left(y = f(x) \rightsquigarrow y = f(g(t)) \rightsquigarrow \frac{dy}{dt} = \underset{\text{"outside"}}{\frac{dy}{dx}} \cdot \underset{\text{"inside"}}{\frac{dx}{dt}} \right)$$

Now, assume $z = z(x,y) \overset{(OR)}{=} f(x,y)$. What if x and y are both functions of t? Then

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$$

or

$$\left(\frac{dz}{dt} = f_x \cdot \frac{dx}{dt} + f_y \frac{dy}{dt} \right) \quad \text{OR} \quad \left(\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \right)$$

if f is confusing

This is saying "the rate of change of z with respect to time" is "how fast x changes with time multiplied by how fast z changes with x" plus "how fast y changes with time multiplied by how fast z changes with y".

EXAMPLE 1

$z = x^3y^2 + 7y^3x$, where $x = \sin(t)$ and $y = \cos t$.

Find $\frac{dz}{dt}$ when $t=0$.

STEP 1 Use chain rule to find $\frac{dz}{dt}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (3x^2y^2 + 7y^3) \cdot (\cos t) + (2x^3y + 21y^2x)(-\sin t)$$

STEP 2 (plug in!) $t=0 \Rightarrow$ $x = \sin(0) = 0$, $y = \cos(0) = 1$

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+7)\cdot 1 + (0)\cdot(0)$$

$$= 7.$$

Notice: If we take a derivative with respect to a variable, and the function really only depends on that one variable, it is a normal derivative, and we would write "$\frac{dz}{dt}$".

If we take a derivative with respect to a variable, and the function depends on multiple variables, then it is a partial derivative, and we would write "$\frac{\partial z}{\partial t}$" or "$z_t$".

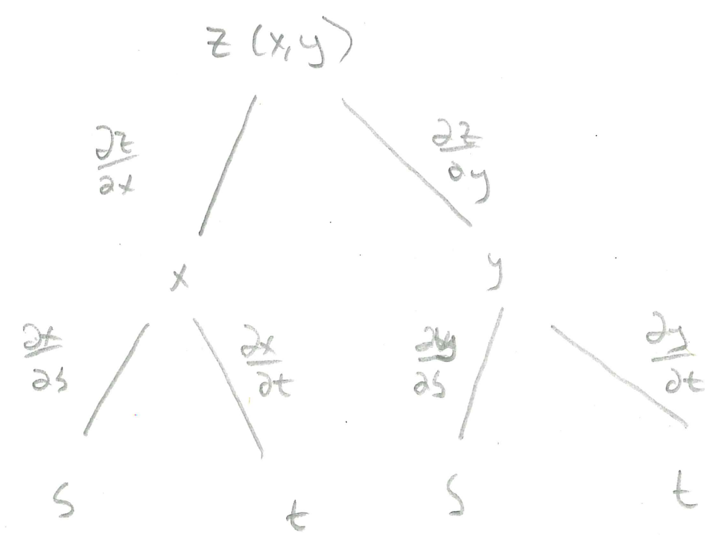
Exercise Explain the use of "normal" derivatives and partial derivatives in the version of the chain rule above. Hint: in the example, plug in the functions of $x(t)$ and $y(t)$ directly into z. Then see that $z(x,y) = z(t)$. ... Do you see how it is a matter of "perspective"?

Now, what if $z = z(x,y) \left(= f(x,y)\right)$, and both x and y are functions of two variables, s and t? We end up with two possible derivatives for z:

$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} \end{aligned}}$$

Notice: We can think about this similarly to the first version of the chain rule: "the rate of change of z with respect to s" is "how fast z changes with x times how fast x changes with s" plus "how fast z changes with y times how fast y changes with s", and similarly for t.

There is also a nice picture in the text!



In fact, we can generalize this case to n-variables with m-variables each, i.e. Let $z = z(x_1, \dots, x_n)$, where $x_1 = x_1(t_1, \dots, t_m)$, $\dots$, $x_n = x_n(t_1, \dots, t_m)$.

Then,

$$\frac{\partial z}{\partial t_1} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_1} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_1} + \cdots + \partial z$$

$$\vdots$$

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_m}$$

$$\vdots$$

$$\frac{\partial z}{\partial t_m} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_m} + \cdots + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_m}.$$

This is the <u>general</u> version of the chain rule. In most cases in our class, we will be working with z as a function of x and y, and x and y functions of possibly two more variables. But you should understand how to do this with any number of variables. We will do a few exercises with higher numbers of variables so you get used to it.

<u>EXAMPLE 2</u>

$u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$, and

$z = r^2 s \sin t$.

Find $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$.

STEP 1: Notice $u(x,y,z)$, $x(r,s,t)$, $y(r,s,t)$, $z(r,s,t)$.

STEP 2: Use general chain rule.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$\frac{\partial u}{\partial z}$

$$= (4x^3y)\cdot(re^t) + (x^4 + 2yz^3)\cdot(2rse^{-t}) + (3y^2z^2)(r^2 \sin t)$$

at $r=2,\ s=1,\ t=0$: $x=2$, $y=2$, $z=0$

$$\left.\frac{\partial u}{\partial s}\right|_{(2,1,0)} = (4\cdot 2^3\cdot 2)(2) + (2^4 + 2\cdot 1\cdot 0^3)(2\cdot 2\cdot 1\cdot 1) + (3\cdot 2^2\cdot 0^2)\cdot(0)$$

$$= 128 + 64 + 0$$

$$\boxed{= 192}$$