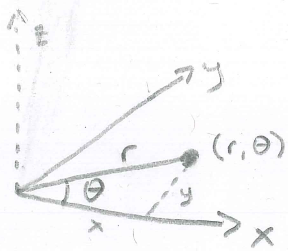
Lecture #7

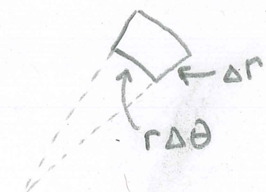
15.7 / 15.8 Triple Integrals in Cylindrical and Spherical Coordinates

REM (Polar Coordinates)

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



Small area:

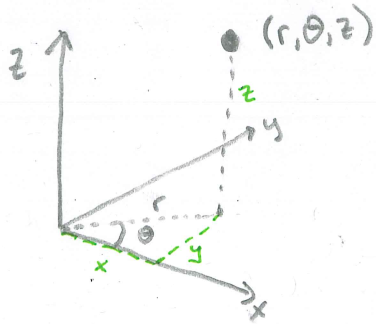


$$\boxed{dA = r\,dr\,d\theta}$$

Cylindrical coordinates and spherical coordinates "extend" the idea of polar coordinates into the 3rd Dimension.

DEF Cylindrical Coordinates

Think: "polar with z as a height"



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$$

NOTICE We put x and y in polar coordinates and leave z untouched.

EXERCISES

① Plot the points whose cylindrical coordinates are given:

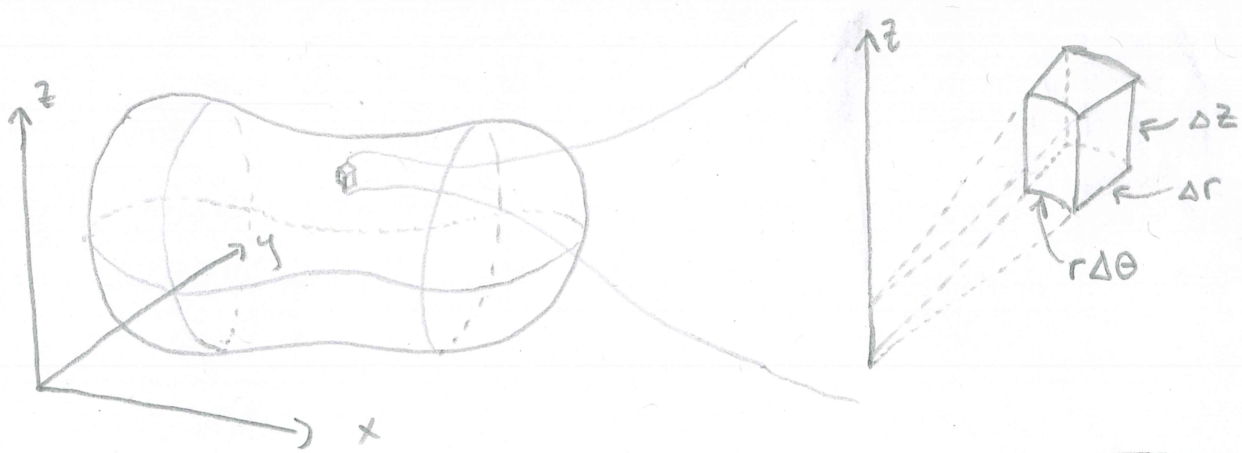
- $(4, \pi/3, -2)$
- $(2, -\pi/2, 1)$
- $(1, 1, 1)$
- $(\sqrt{2}, 3\pi/4, 2)$

what are the corresponding rectangular coordinates?

② Change the following points from rectangular to cylindrical coordinates:

- $(-1, 1, 1)$
- $(-\sqrt{2}, \sqrt{2}, 1)$
- $(-2, 2\sqrt{3}, 3)$

QUESTION Given a volume in 3D space, how do we integrate over this volume in cylindrical coordinates?



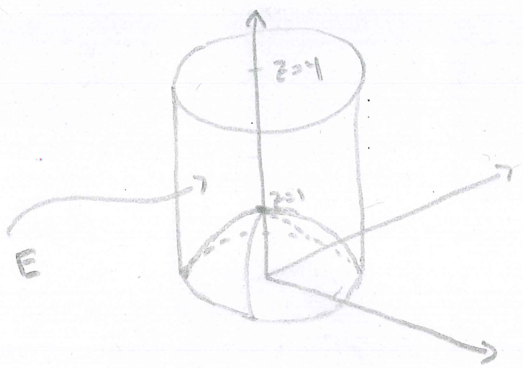
$$\boxed{dV = r\,dr\,d\theta\,dz}$$

Think: "$\Delta V \approx \underbrace{r\Delta\theta\Delta r}_{\text{polar}} \cdot \Delta z$. In the limit $\Delta V \to dV$ and $r\Delta\theta\Delta r\Delta z \to r\,d\theta\,dr\,dz$."

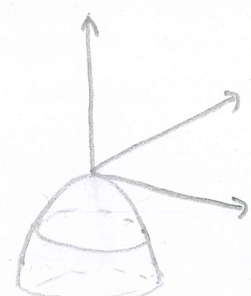
EXAMPLE 1

A solid lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

STEP 1: Identify the solid E.



(Aside: paraboloid $z=-(x^2+y^2)$ this is the paraboloid facing down!



So, $z=-x^2-y^2+1$ is just this same paraboloid, but moved up one unit.)

STEP 2: Write equations of surfaces in cylindrical coordinates.

$$z=4 \implies z=4$$

$$1=x^2+y^2 \implies 1=r^2, \text{ so } r=1 \quad (*)$$

$$\underbrace{z=1-x^2-y^2}_{\text{rectangular}} \implies \underbrace{z=1-r^2}_{\text{cylindrical}}$$

STEP 3: Density function?

distance from z-axis (axis of the cylinder)

$$\rho(r,\theta,z) = K \cdot \sqrt{x^2+y^2} = Kr.$$

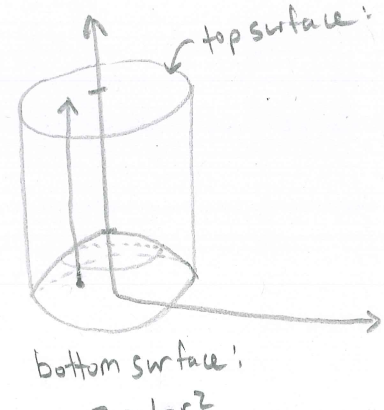
STEP 4: Set-up Integral! (Choose order of integration!)

$$\iiint_E \rho(r,\theta,z)\,dV = \iiint_E Kr\,dV$$

We need to choose a variable to integrate first...

REMARK Usually, you will want to integrate the "non-polar" coordinate first, i.e. the z coordinate.

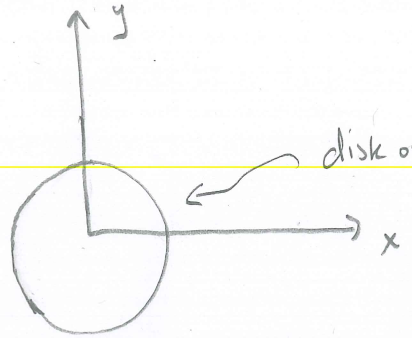
We'll try z:



$$= \underbrace{\iint}_{\uparrow} \int_{z=1-r^2}^{z=4} (Kr)\underbrace{r\,dz\,dr\,d\theta}_{(dV)}$$

Project the region onto the xy-plane. Then ask "What values of (x,y) does 'starting at $z=1-r^2$' and 'ending at $z=4$' make sense?"

projection:



disk of radius 1 ($x^2+y^2 \le 1$ or $r \le 1$)

NOTE For any (x,y) in the disk, the integration scheme for z works, (or makes sense) !

polar bounds!
$$\begin{bmatrix} r=0 \longmapsto r=1 \\ \theta=0 \longmapsto \theta=2\pi \end{bmatrix}$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=1-r^2}^{z=4} (Kr^2)\, dz\, dr\, d\theta$$

extra r from dV!

STEP 5: Compute!

$$\int_0^{2\pi}\int_0^1\int_{1-r^2}^4 Kr^2\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 Kr^2\left(4-(1-r^2)\right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(4Kr^2 - Kr^2 + Kr^4\right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(3Kr^2 + Kr^4\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left[Kr^3 + \frac{Kr^5}{5}\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(K + \frac{K}{5} - 0\right) d\theta$$

$$= \frac{6K}{5}\int_0^{2\pi} d\theta = \frac{6K}{5}\cdot 2\pi = \boxed{\frac{12\pi K}{5}}$$

EXERCISES

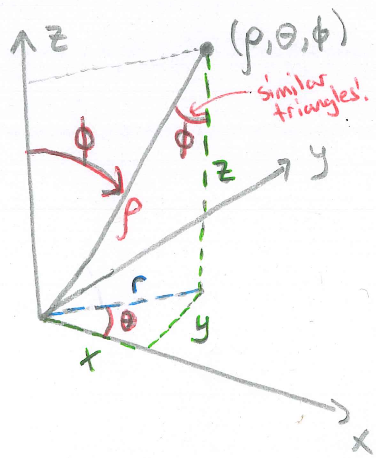
① Try Example 4 in 15.7.

② Evaluate the triple integrals by changing to cylindrical coordinates:

a) $$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy,$$

b) $$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\,dz\,dy\,dx.$$

DEF Spherical Coordinates



$$\boxed{\rho \geq 0,\quad 0 \leq \phi \leq \pi,\quad 0 \leq \theta \leq 2\pi}$$

NOTICE
$z = \rho\cos\phi$
$r = \rho\sin\phi$

Then
$x = r\cos\theta = \rho\sin\phi\cos\theta$
$y = r\sin\theta = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

So the transformation from spherical to rectangular coordinates:

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \qquad \begin{cases} \rho^2 = x^2+y^2+z^2 \\ \text{[Don't worry about } \theta,\phi \text{ right now..]} \end{cases}$$

NOTICE In spherical coordinates, a sphere of radius 1, $x^2+y^2+z^2=1$, has the equation $\rho = 1$.

EXERCISES

① Convert from spherical to rectangular coordinates:

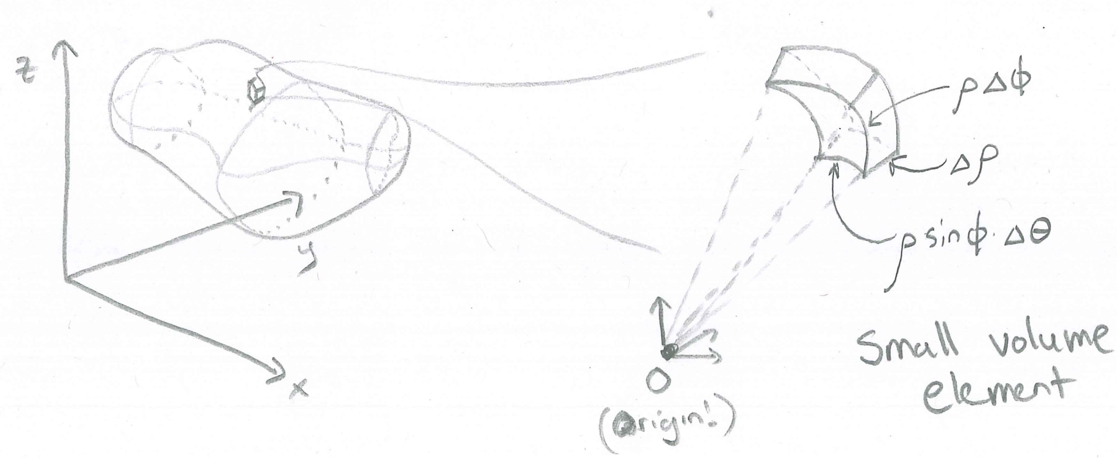
$(6, \pi/3, \pi/6)$, $(4, -\pi/4, \pi/3)$, $(2, \pi/2, \pi/2)$

② Convert from rectangular to spherical coordinates:

$(0,-2,0)$, $(\sqrt{3}, -1, 2\sqrt{3})$, $(1, 0, \sqrt{3})$

QUESTION

Given a volume in 3D space, how do we integrate over this volume in spherical coordinates?



NOTE The bottom arc in the "small volume element" is actually $r \cdot \Delta\theta$, but here $r = \rho\sin\phi$, so we see it as $\rho\sin\phi\,\Delta\theta$. Similarly, the arc labeled $\rho\Delta\phi$ is the piece of a circle of radius ρ, where the change in angle is ϕ (similar to the $r\,\Delta\theta$ in polar coordinates).

So $dV \approx \rho\sin\phi \cdot \Delta\theta \cdot \rho\Delta\phi \cdot \Delta\rho$, i.e. taking a limit, we see

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$

EXAMPLE 2 Use spherical coordinates to find the volume of the solid that lies above $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.

STEP 1: Identify the solid.

Notice, we are told $x^2+y^2+z^2=z$ is a sphere. What is the center? radius? Put it in standard form by completing the square:

$$x^2+y^2+z^2-z=0$$

$$x^2+y^2+\left(z^2-z+\tfrac{1}{4}\right)\underbrace{-\tfrac{1}{4}}_{\text{add }0}=0$$

$(z+a)^2=z^2+2az+a^2$

$\Rightarrow 2a=-1,\ a=\frac{-1}{2}$

$\Rightarrow a^2=\frac{1}{4}$

$$x^2+y^2+\left(z^2-z+\tfrac{1}{4}\right)=\tfrac{1}{4}$$

$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2=\tfrac{1}{4}$$

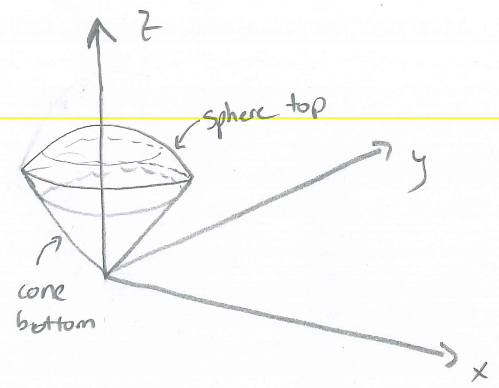
Center: $(0,0,\frac{1}{2})$

radius: $\sqrt{\frac{1}{4}}=\frac{1}{2}$

Next, we need to recognize $z=\sqrt{x^2+y^2}$ as a cone, or rather, the top half of the cone $z^2=x^2+y^2$ (the bottom half being $z=-\sqrt{x^2+y^2}$).

[EXERCISE If you do not recognize this as a cone, take traces and draw the cone.]

Then, the region is below the sphere, and above the cone:



STEP 2: Write equations in spherical coordinates.

sphere:

$$\underbrace{x^2+y^2+z^2=z}_{\text{rectangular}} \longmapsto \underbrace{\begin{gathered}\rho^2=\rho\cos\phi \\ \boxed{\rho=\cos\phi}\end{gathered}}_{\text{spherical}}$$

cone:

$$z=\sqrt{x^2+y^2} \longmapsto \rho\cos\phi=\sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}$$

$$\rho\cos\phi=\sqrt{\rho^2\sin^2\phi\,(\cos^2\theta+\sin^2\theta)}$$

$$\rho\cos\phi=\rho\sin\phi$$

$$\cos\phi=\sin\phi,$$

$$\boxed{\phi=\frac{\pi}{4}}$$

$\boxed{\text{rem}}$ (*) $0\le\phi\le\pi$

only one solution!

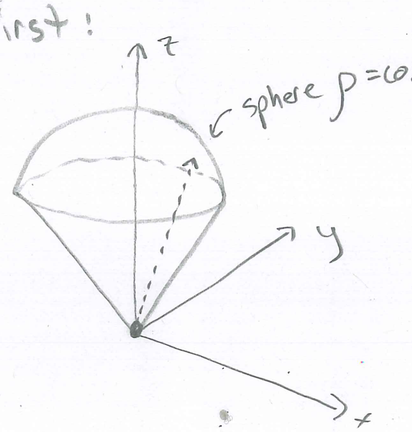
rectangular spherical

(So, the equation of the cone is $\phi=\frac{\pi}{4}$. Go back to the definition of spherical coordinates and convince yourself this works!)

STEP 3: Set-up the integral! (Choose order of integration.)

$$\text{Volume} = \iiint_E dV = \iiint_E \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

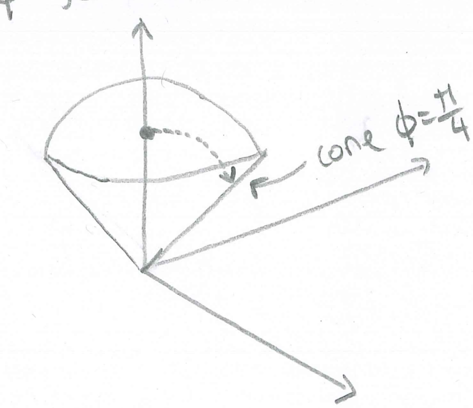
Choose ρ first!



start at origin, integrate to the top surface $\rho = \cos\phi$

$$= \iint \int_{\rho=0}^{\rho=\cos\phi} \rho^2 \sin\phi \, d\rho \, (d\phi \, d\theta)$$

Choose ϕ second:



start at $\phi = 0$ and go until you hit the cone $\phi = \pi/4$

And integrate θ in a full circle, from 0 to 2π.

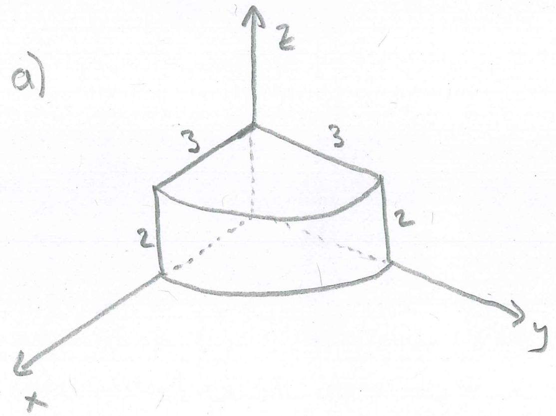
$$= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/4} \int_{\rho=0}^{\rho=\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

STEP 4: Compute! (You should get $\frac{\pi}{8}$.)

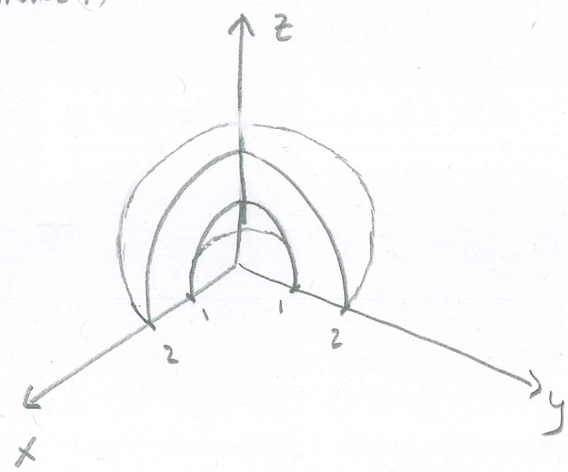
EXERCISES

① Set up the triple integral of $f(x,y,z)$ over the regions below:
(use cylindrical or spherical coordinates!)

a)



b)



(This is a sphere of radius two with a sphere of radius one removed, the bottom half removed, and then the piece in the first octant removed.)

② Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\cdot e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = 2\pi.$$

Hint: Define the improper integral the limit of a triple integral over solid spheres as the radius of the sphere increases indefinitely.

③ Evaluate the integrals by changing to spherical coordinates:

a) $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$ b) $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx$