Lecture #7

15.10 (Webassign) / 15.9 (Book) Change of Variables in Multiple Integrals

Rem u-sub

$$\int_0^{\pi} \sin(x^2)\cdot 2x\,dx = \int_{u(0)=0}^{u(\pi)=\pi^2} \sin(u)\,du$$

u-sub

$$u = x^2$$
$$du = 2x\,dx$$

Now, lets reverse the role of x and u:

$$\int_0^{\pi^2} \sin(x)\,dx = \int_0^{\pi} \sin(u^2)\cdot 2u\,du$$

$$x = u^2 \leftarrow g(u) \text{ or } x(u)$$

$$dx = \underbrace{2u}_{g'(u) \text{ or } x'(u)}du$$

NOTICE In this form, we can write a formula that says precisely how to change coordinates from x-coordinates to u-coordinates.

$$\int_{a=g(c)}^{b=g(d)} f(x)\,dx = \int_{g^{-1}(a)=c}^{g^{-1}(b)=d} f(\underbrace{g(u)}_{x(u)^*})\cdot \underbrace{g'(u)}_{\text{or } x'(u)^*}\,du$$

If you do not like the "g" notation (where $g(u)$ is u^2 in the example) and you would rather think of $x(u) = \underline{x = u^2}$, i.e. $x(u) = u^2$, then we could write the following:

$$\int_{a = x(a)}^{b = x(b)} f(x)\,dx \;=\; \int_{c = x^{-1}(a)}^{d = x^{-1}(b)} f(x(u))\,x'(u)\,du$$

Both formulas are identical! ~~One might suggest method~~

<u>REMARK</u> This equation, coming from the idea of u-substitution, gives us a way to change from x-coordinates to u-coordinates in the one dimensional case.

Our goal is to extend this idea into n-dimensions! We'll start with 2 though.

<u>REM</u> Polar coordinates! We have already done this!

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

(but which one? ... lots of derivatives

And integration:

$$\iint_R f(x,y)\,dx\,dy \;=\; \iint_{\underbrace{S}_{①}} f(r\cos\theta, r\sin\theta)\cdot \underbrace{r}_{②}\,dr\,d\theta$$

① S is Rewritten in polar coordinates.

② this has something to do with a derivative!

We will focus on ① for a moment, and hold off on the derivative issue (②).

If we change coordinates to something other than polar, we need to know how to re-write the region R in the new coordinates.

EXAMPLE 1

Consider the change of coordinates defined by

$$x = u^2 - v^2$$

$$y = 2uv.$$

Find the image of the square $S = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$.

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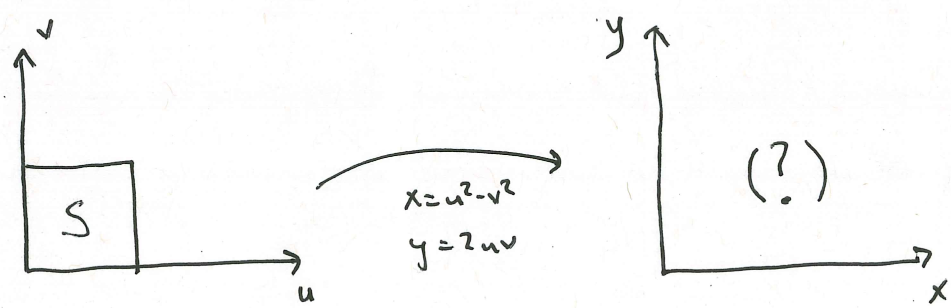
NOTICE We are going backwards! S is given in the u-v plane:



↑ integrating here is: ↑ integrating here is:

$$\iint_S f(x(u),y(u)) \cdot \text{some derivative thing} \cdot du\,dv = \iint_{(?)} f(x,y)\,dx\,dy$$

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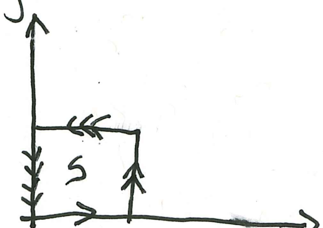
To find the image of S under $x=u^2-v^2$ and $y=2uv$, we need to figure out where the edges of S go, so lets label them:

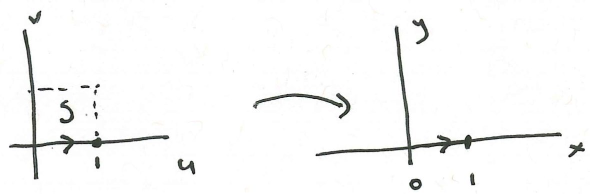


Start with $\rightarrow$, where $v = 0$, $0 \le u \le 1$. Plug it into our equation!

$$x = u^2 - 0 = u^2$$

$$y = 0$$

So, along this first edge, it gets mapped to $x = u^2$ from 0 to 1, where $y = 0$.



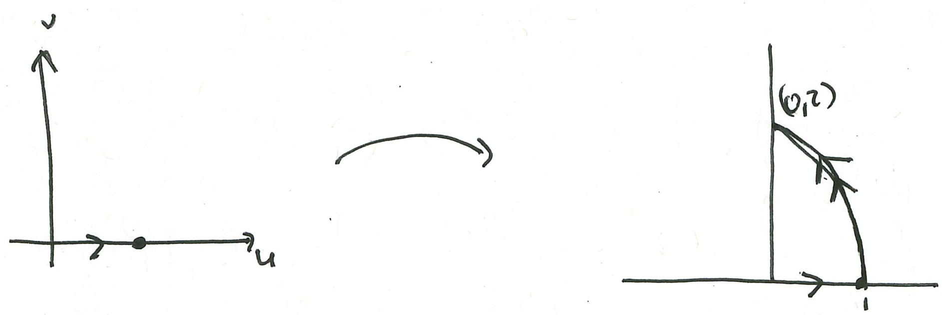
Next, lets do $\uparrow$, where $u = 1$, $0 \le v \le 1$. Plug in:

$$\left.\begin{aligned} x &= 1 - v^2 \\ y &= 2v \end{aligned}\right\}$$ system of equations! solve $x(y)$ or $y(x)$.

$(\frac{1}{2}y = v)$

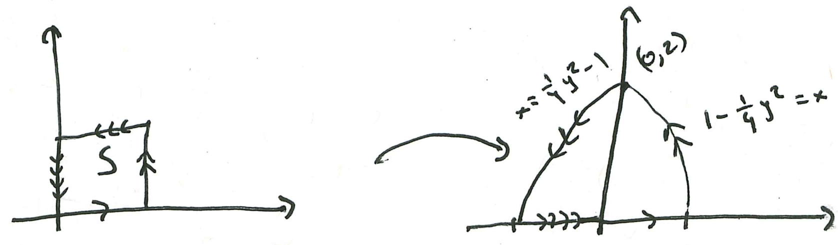
$$x = 1 - \left(\frac{1}{2}y\right)^2$$

$$x = 1 - \frac{y^2}{4}$$



<u>Exercise</u> do this for the other two sides!

<u>Solution</u>:

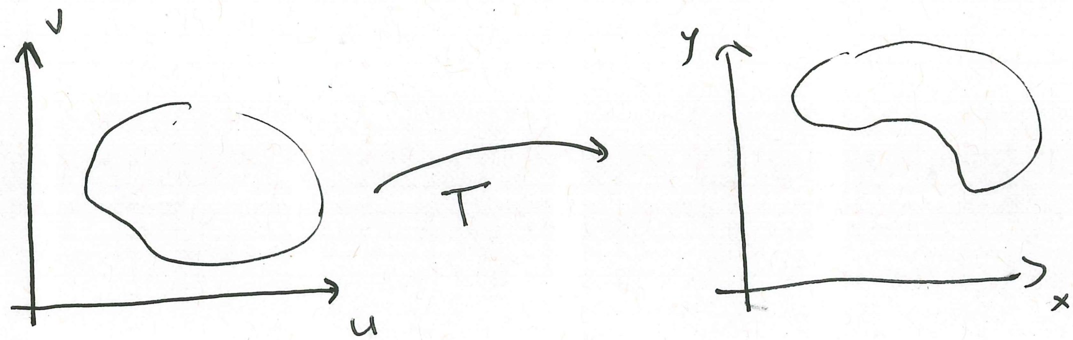


REMARK We typically think of

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

as a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, i.e.

$T(u,v) = (x(u,v), y(u,v))$, where

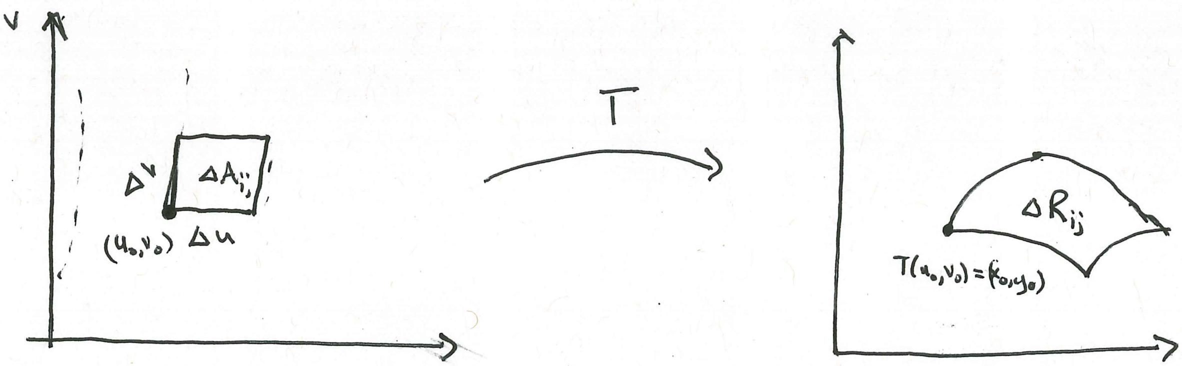


T maps from the u,v-plane to the xy-plane.
(*) (*)

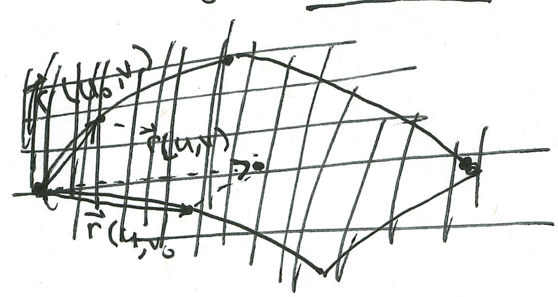
For this to work, we need T to be a C^1-transformation, in other words, it needs to have all partial derivatives of order 1, and the map must be one-to-one onto the image. This property means there is an inverse transformation, which we want! (Why?) More to the point, we really want the map to be one-to-one ... otherwise to 'coordinates' in the uv-plane get sent to the same 'coordinate' in the xy-plane. (What value of the function do you assign to this point when you integrate?)

Now, we have to figure out ② ... ~~understand~~ how the derivative plays a role.

To understand this, we need to understand how a small piece of area $\Delta A = \Delta u \cdot \Delta v$ in the uv-plane produces a small piece of area in the xy-plane under the change of coordinates (or transformation).



Now, look closely at ΔR_{ij}. Can we define a function along boundary? Yes! By defining a vector function on the region

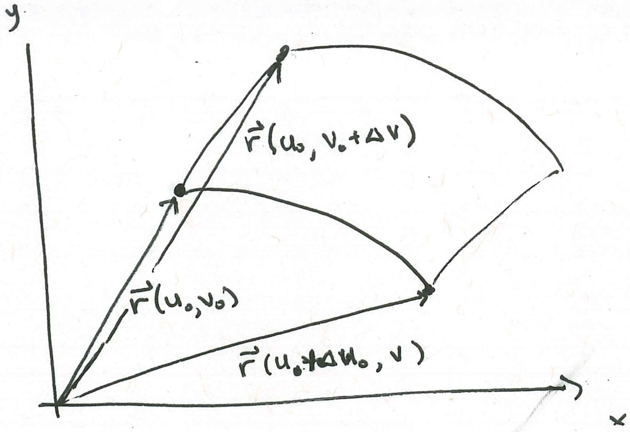


vector valued!

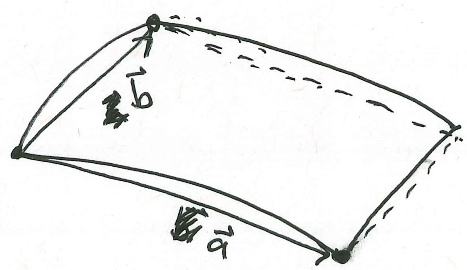
$$T(u,v) \rightsquigarrow \vec{r}(u,v) = \underbrace{g(u,v)}_{\text{or } \to\, x(u,v)}\,\hat{i} + \underbrace{h(u,v)}_{\text{or } \to\, y(u,v)}\,\hat{j}$$

(this is the coordinate transformation!)

$\vec{r}$:



Notice, as ΔR_{ij} gets small (when $\Delta u \cdot \Delta v$ gets small), we can approximate the area of ΔR_{ij} by computing the area of the parallelogram : $|\vec{a} \times \vec{b}|$, where



$$\vec{b} = \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0,v_0)$$

$$\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0)$$

Notice, as $\Delta u \to 0$ and $\Delta v \to 0$, we can actually say what $\vec{a}$ and $\vec{b}$ will be :

$$\lim_{\Delta u \to 0} \frac{\vec{a}}{\Delta u} = \frac{\vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0)}{\Delta u} = \vec{r}_u$$

partial w.r.t. u !

$$\lim_{\Delta v \to 0} \frac{\vec{b}}{\Delta v} = \frac{\vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0,v_0)}{\Delta v} = \vec{r}_v$$

partial w.r.t. v !

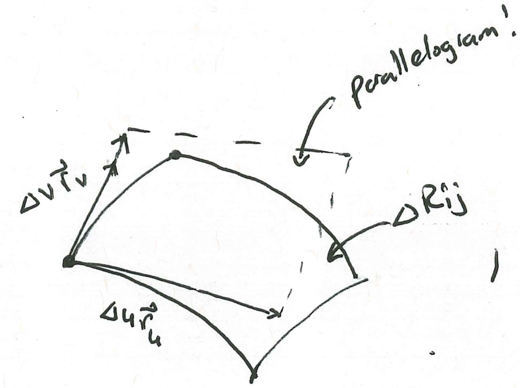
So, as Δu and Δv are getting very small ...

$$\vec{a} = \vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0) \approx \Delta u \cdot \vec{r}_u$$

$$\vec{b} = \vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0) \approx \Delta v \cdot \vec{r}_v$$

i.e.

$\approx$



, $\Delta R_{ij} \approx |\Delta u \vec{r}_u \times \Delta v \vec{r}_v|$

Then, to approximate ΔR_{ij}, we want to compute:

$$|(\Delta u \cdot \vec{r}_u) \times (\Delta v \cdot \vec{r}_v)| = |\vec{r}_u \times \vec{r}_v| \cdot \underbrace{\Delta u \Delta v}_{\text{scalars}} \quad (*)$$

Then

$$\vec{r}_u = \underbrace{\frac{\partial x}{\partial u}}_{\text{or } \frac{\partial g}{\partial u}} \hat{\imath} + \underbrace{\frac{\partial y}{\partial u}}_{\text{or } \frac{\partial h}{\partial u}} \hat{\jmath} \quad \leftarrow \text{depending on notation you like.}$$

$$\vec{r}_v = \frac{\partial x}{\partial v} \hat{\imath} + \frac{\partial y}{\partial v} \hat{\jmath},$$

So,

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\text{determinant of a 2x2 matrix.}} \hat{k}$$

magnitude is just the scalar here!

and
$$|\vec{r}_u \times \vec{r}_v| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{} & \frac{\partial y}{} \end{vmatrix}$$

DEF We say the Jacobian of a transformation is

Notice, we can swap the ~~vector~~ positions of $\frac{\partial x}{\partial v}$ and $\frac{\partial y}{\partial u}$ without changing the determinant!

$$\cancel{|}J\cancel{|} = \cancel{|}\frac{\partial (x,y)}{\partial (u,v)}\cancel{|} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}.$$

ignore!

NOTE Typically, you call the 2x2 matrix J the Jacobian, ~~and~~ Above, we are computing the <u>determinant</u> of this matrix J.

Then, we see that

absolute value of the determinant $= \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right|$

$$\Delta A \simeq |J| \cdot \Delta u \cdot \Delta v, \text{ i.e.}$$

$$dxdy = \boxed{dA = |J| \cdot du \cdot dv.}$$

Exercise Compute the Jacobian for the polar coordinate transformation.

Now, we can finally say what we mean by change of variables in higher dimension

DEF Change of Variable in a Double Integral

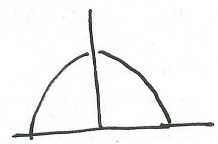
$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \cdot \left| \frac{\partial (x,y)}{\partial (u,v)} \right| du dv.$$

Notice we need the Jacobian to be <u>non-zero</u>, i.e. $T(u,v) = (x(u,v), y(u,v))$ must be an invertible transformation!

EXAMPLE 2

Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

① R:



← Same as image of square in example 1! (show this!)

② Compute Jacobian:

$$|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = \left| \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} \right| = |4u^2 + 4v^2| \geq 0$$

$$= 4u^2 + 4v^2$$

③ Compute integral:

$$\iint_R y\,dA = \iint_S (2uv) \cdot |J|\,du\,dv = \int_0^1\int_0^1 2uv\,(4u^2 + 4v^2)\,du\,dv$$

$$= 8 \int_0^1\int_0^1 (u^3 v + uv^3)\,du\,dv$$

$$= 8 \int_0^1 \left[\tfrac{1}{4}uv + \tfrac{1}{2}u^2v^3\right]_0^1 dv$$

$$= 8 \int_0^1 \left(\tfrac{1}{4}v + \tfrac{1}{2}v^3\right) dv$$

$$= 8 \left(\left[\tfrac{1}{8}v^2 + \tfrac{1}{8}v^4\right]_0^1\right)$$

$$= 2.$$