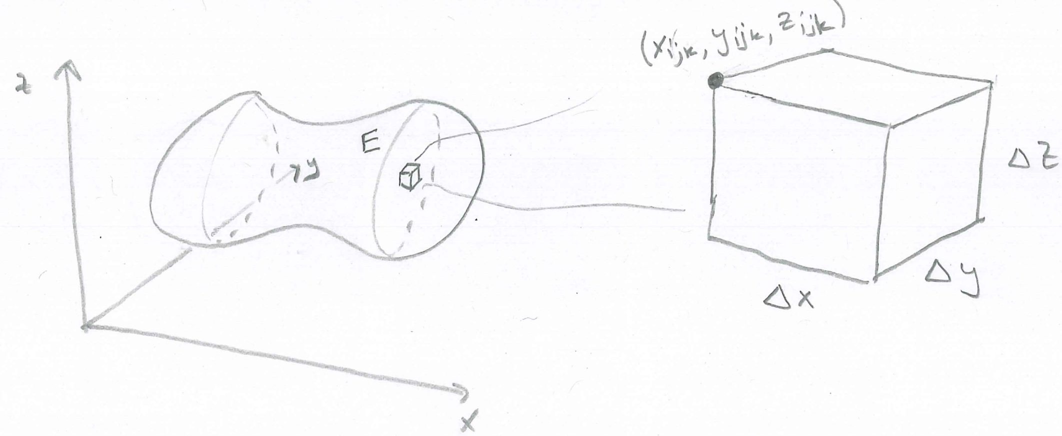
Lecture #6

15.6 Triple Integrals

DEF
$$\iiint_E f(x,y,z)\, dV := \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}, y_{ijk}, z_{ijk}) \cdot \Delta V$$

In Cartesian coordinates, $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$:



Think: "$f(x,y,z)$ is an assigment of a value at each point in E."
For example, f may assign a temperature at each point.

REMARK Provided f is continuous and the region E is bounded, we can integrate with respect to any variable first! In fact, there are 6 different ways to order the integral:

- $dx\,dy\,dz$
- $dx\,dz\,dy$
- $dy\,dx\,dz$
- $dy\,dz\,dx$
- $dz\,dx\,dy$
- $dz\,dy\,dx$

(Fubini for triple integrals!)

NOTE The region E need not be bounded, but this makes the situation more delicate.

EXAMPLE 1

Evaluate $\iiint_B e^z \, dV$ where $B = \{(x,y,z) \mid 0 \le x \le 1, -1 \le y \le 3, 0 \le z \le 2\}$.

this is a box! (Draw a picture!)

$$\iiint_B e^z \, dV = \int_0^2 \int_{-1}^3 \int_0^1 e^z \, dx \, dy \, dz$$

$$= \int_0^2 e^z \left[\int_{-1}^3 \int_0^1 dx \, dy \right] dz$$

$$= \int_0^2 e^z \left[\int_{-1}^3 [x]_0^1 \, dy \right] dz$$

$$= \int_0^2 e^z \left[\int_{-1}^3 dy \right] dz$$

$$= \int_0^2 e^z \, [y]_{y=-1}^{y=3} \, dz$$

$$= \int_0^2 4e^z \, dz$$

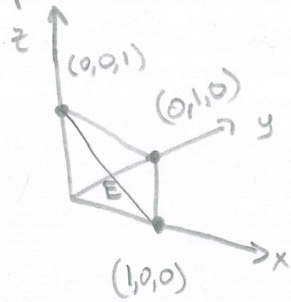
$$= 4 \left[e^z \right]_{z=0}^{z=1}$$

$$= \boxed{4e^2 - 4} \overset{(OR)}{=} 4(e^2 - 1).$$

EXAMPLE 2

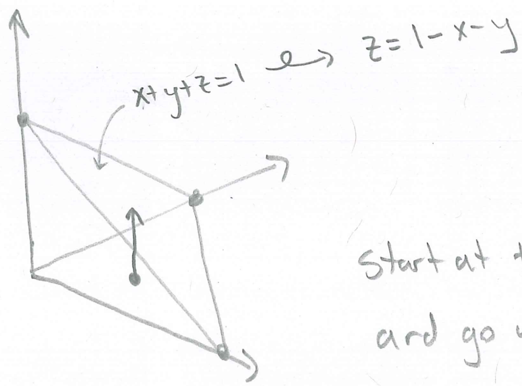
Evaluate $\iiint_E z \, dV$, where E is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$, and $x+y+z=1$.

STEP 1 Draw a picture! Plug in points ... intersect $x+y+z=1$ with $\begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$



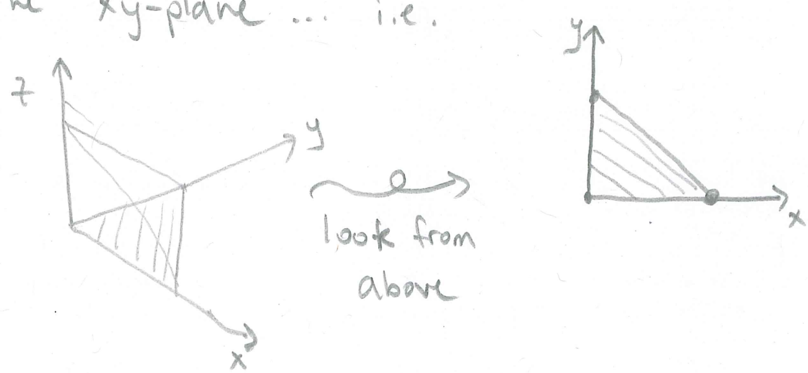
... and identify the bounded region.

STEP 2 Pick a direction to integrate. Let's try to start with z,



Start at the bottom, $z=0$, and go until you hit the top, $z=1-x-y$.

Question: For any (x,y) in the shadow of E on the xy-plane ... i.e.



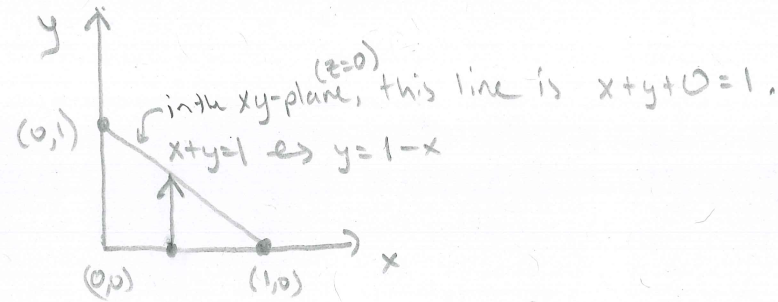
does z go from $z=0$ to $z=1-x-y$?

Yes! So... we can start setting up the integral

$$\iiint_E z\,dV = \iint \int_{z=0}^{z=1-x-y} z\cdot dz \;\; d?\,d?$$

STEP 3 Pick the next direction to integrate. Let's try y.

(Notice, we are looking at the "shadow".)



Notice, y goes from 0 to $1-x$. And this will work for x between 0 and 1.

Notice, for the last two integration variables, this is just like the set-up of a double integral.

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} z\;dz\,dy\,dx$$

STEP 4 Compute!

EXERCISE ① Compute this integral! You should get $\frac{1}{24}$.

② Set-up this integral in two more (different) ways.

EXAMPLE 3

Rewrite the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, dz\, dy\, dx$ as an iterated integral in a different order.

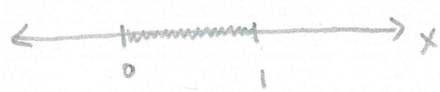
STEP 1 Figure out the domain!

Two ways : ① start outside and work in

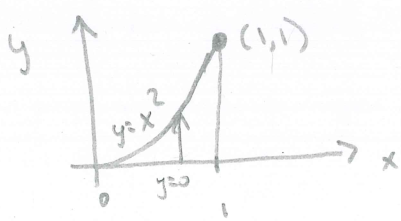
② start inside and work out

① seems easiest to me :

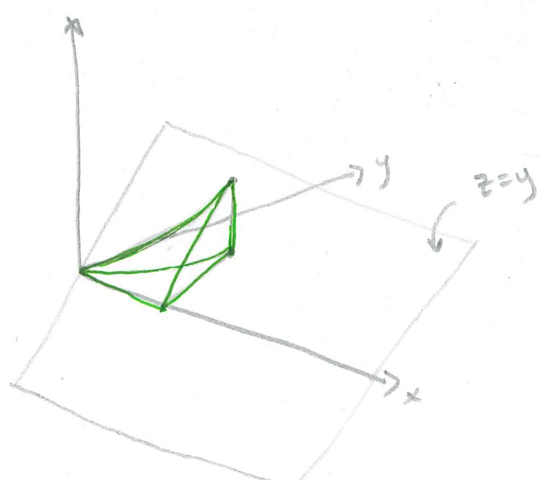
x : must be between 0 and 1



y : must be between 0 and x^2

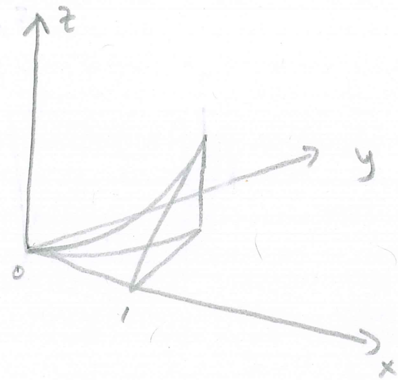


z : must be between 0 and y. Note, z=0 and z=y are planes!



(z=0 is the xy-plane)

So, our region looks like



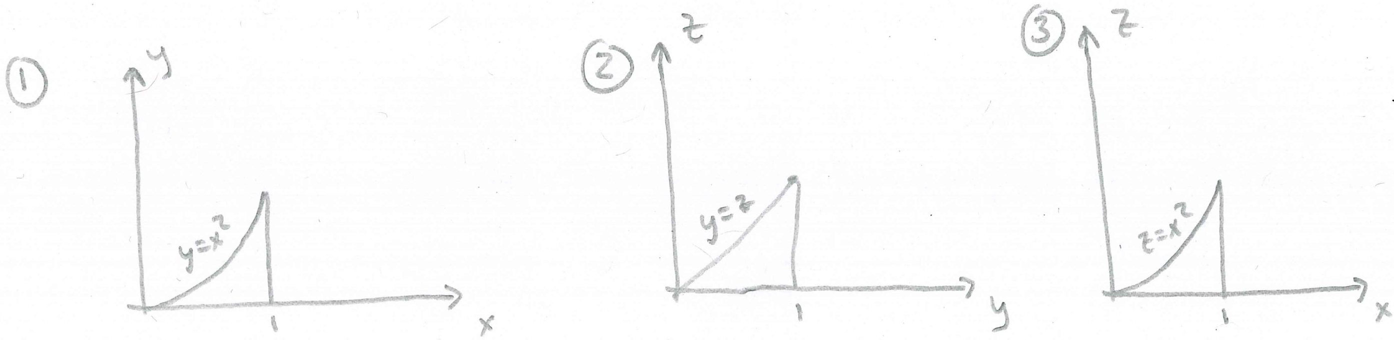
<u>STEP 2</u>

Consider projections of the region onto each of the planes:

① xy-plane

② yz-plane

③ xz-plane



<u>QUESTION</u> Why consider the projections? These are the "shadows".

If you choose to integrate z first, you'd need to consider ① to write the last two bounds. If you choose to integrate x first, you need to consider ②. Similarly, if you choose to integrate y first, you need to consider ③.

STEP 3 Choose an order of integration. We'll try

$$\cdot \, dx \, dz \, dy$$

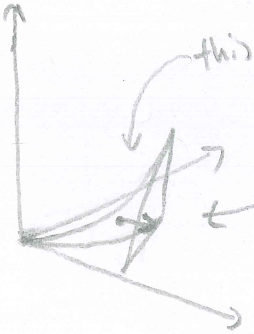
NOTE This one is more challenging than other options ... try some others!

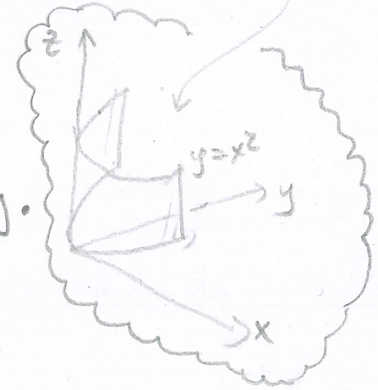
Integrating x first:

this side is $y=x^2$ (left side)

NOTE Think of this in 3-d.

this side is $x=1$ (right side)



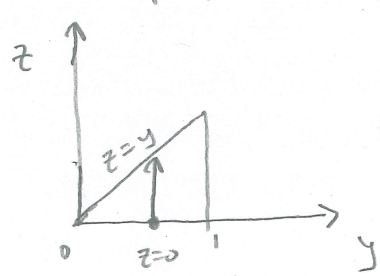


Notice, $y=x^2 \Leftrightarrow x=\pm\sqrt{y}$, and we want $x=\sqrt{y}$.

Then, x goes from $x=\sqrt{y}$ to $x=1$

$$\iint \left[\int_{x=\sqrt{y}}^{x=1} f(x,y,z)\, dx \right] dz\, dy$$

Then, look at the projection



and integrate over this region. Notice z goes from 0 to y while y goes from 0 to 1.

Thus, we get

$$\boxed{\int_{y=0}^{y=1}\int_{z=0}^{z=y}\int_{x=\sqrt{y}}^{x=1} f(x,y,z)\,dx\,dz\,dy}.$$

EXERCISES ① Do this for two other orders of integration!

② Read through Section "15.6 Triple integrals" starting on p. 1074.

③ Work through Example 6.

REMARK You can find formulas for center of mass, moment about x-axis (or y or z-axis), and moment of inertia for 3-dimensional objects in the textbook. This is the content of example 6 referenced in the exercises above.

(You do not need to know the formulas for 3-dimensional objects, only how to use them. You do, however, need to know the formulas for the 2-dimensional objects covered in 15.4.)

REMARK · We can think of a triple integral as a "4-dim" volume ... or as a way to compute a "3-dim" volume when we integrate

$$\underbrace{\iiint_E dV}_{\text{"sum up little pieces of volume"}} = \iiint_E dx\,dy\,dz .$$

· This is analogous to thinking of double integrals as areas of regions ($\iint_R dA$).

EXERCISES ① Compute the volume of the solid E described in EXAMPLE 2 in these notes.

[Hint: Instead of integrating $\iiint_E z\,dV$, integrate $\iiint_E dV$.]

② Try problem #34 in the book.

③ The average value of a function $f(x,y,z)$ over a solid region E is defined to be

$$f_{avg} = \frac{1}{V(E)} \iiint_E f(x,y,z)\,dV$$

where $V(E)$ is the volume of E. For instance, if ρ is a density function, then ρ_{avg} is the average density of E.

a) Find the average value of the function $f(x,y,z) = xyz$ over a cube with side length L that lies in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

b) Find the average height of the points of the solid hemisphere $x^2+y^2+z^2 \leq 1$, $z \geq 0$.