Lecture #6

Exercises (Triple Integrals)

- The average value of a function $f(x,y,z)$ over a solid region E is defined to be

$$f_{avg} = \frac{1}{V(E)} \iiint_E f(x,y,z)\, dV$$

where $V(E)$ is the volume of E. For instance, if ρ is a density function, then ρ_{avg} is the average density of E.

① Find the average value of the function $f(x,y,z) = xyz$ over a cube with side length L that lies in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

② Find the average height of the points of the solid hemisphere $x^2+y^2+z^2 \leq 1$, $z \geq 0$.

- - - - - - - - - - -

% ~~Let $z=ax+by+c$ be a plane~~.

Exercises (Surface Integrals)

- Let $z=ax+by+c$ be a plane. Show that the area on the plane $z=ax+by+c$ above any region D in the xy-plane is $\sqrt{1+a^2+b^2}\cdot A(D)$.

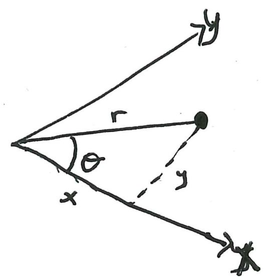
- Alternatively, pick any region D on the plane $z=ax+by+c$ and let $\pi(D)$ be the projection of that region onto the xy-plane. Show that the area of D is $\sqrt{1+a^2+b^2}\cdot A(\pi(D))$.

15.8/15.9 (Webassign), 15.7/15.8 (Book)

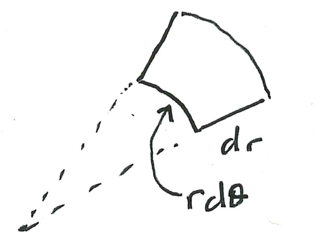
Triple integrals in Cylindrical and Spherical coordinates.

Rem (polar coordinates)

$$\begin{cases} r^2 = x^2+y^2 \\ \tan\theta = \frac{y}{x} \end{cases} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



Small Area:

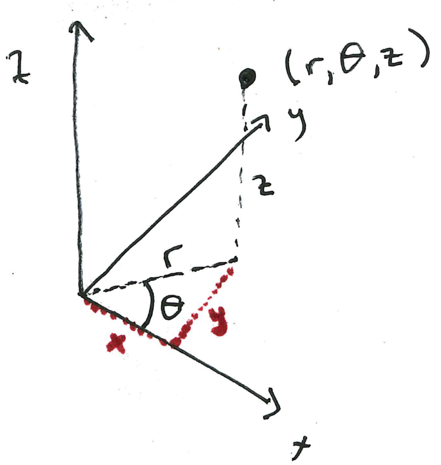


$dA = r\,dr\,d\theta$

Cylindrical coordinates and spherical coordinates "extend" the idea of polar coordinates into the 3rd dimension.

DEF Cylindrical Coordinates

"Think: polar with z as a height"



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r^2 = x^2+y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$$

Notice we put x and y in polar coordinates and leave z untouched.

Exercise Plot the points ~~and~~ whose cylindrical coordinates are given

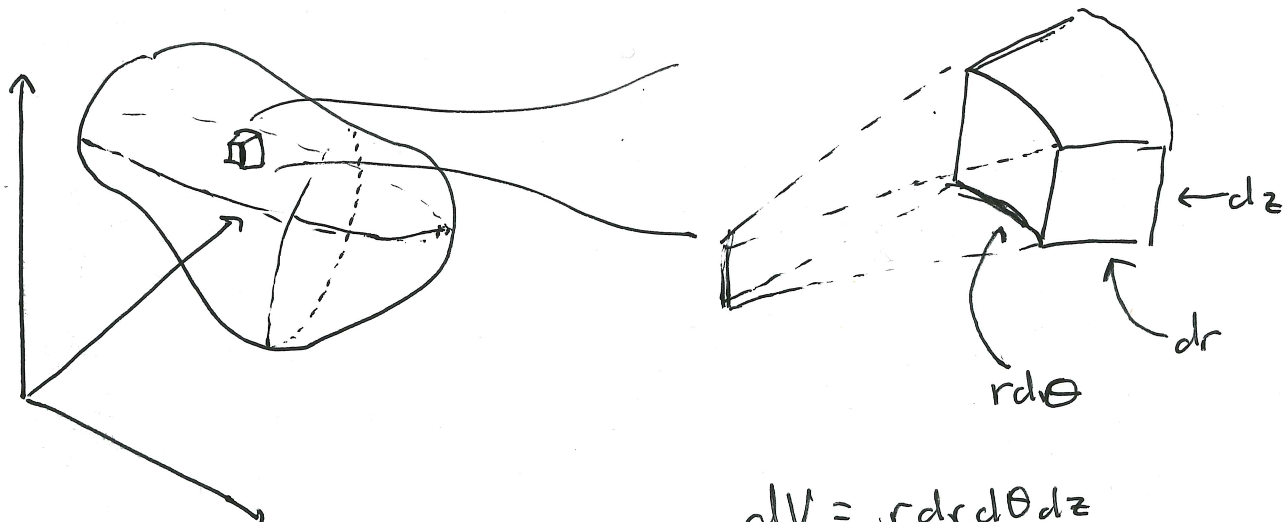
- $(4, \pi/3, -2)$
- $(2, -\pi/2, 1)$
- $(1, 1, 1)$
- $(\sqrt{2}, 3\pi/4, 2)$

what are the rectangular coordinates?

Exercise Change the from rectangular to cylindrical coordinates.

- $(-1, 1, 1)$
- $(-\sqrt{2}, \sqrt{2}, 1)$
- $(-2, 2\sqrt{3}, 3)$

Question Given a volume in 3-D space, ~~what~~ How do we integrate over this volume in cylindrical coordinates?



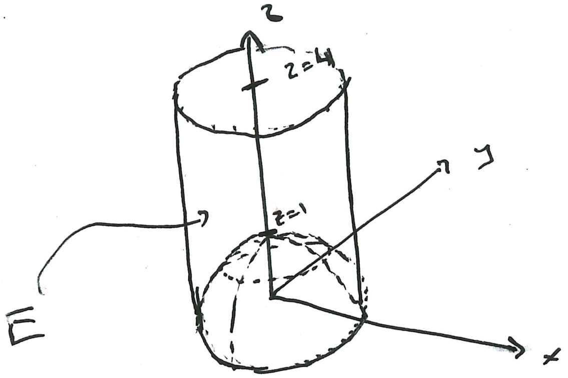
$dV = r\,dr\,d\theta\,dz$

"Think": $\Delta V \approx \underbrace{r\Delta\theta \cdot \Delta r}_{\text{polar}} \cdot \Delta z$. In the $\underline{\underline{\text{limit}}}$, $\Delta V \to dV$

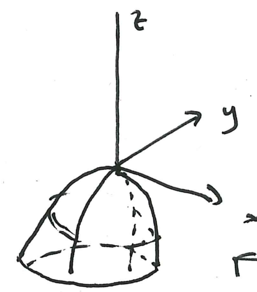
$r\Delta\theta\Delta r\Delta z \to r\,d\theta\,dr\,dz$

EXAMPLE 1 A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Step one : Identify the solid E.



Aside:
paraboloid
$z=-x^2-y^2$



$z=(-x^2-y^2)+1$ moves this up one unit!

Step 2 : Write equations in ~~pol~~ cylindrical coordinates.

$$z=4 \implies z=4$$

$$1=x^2+y^2 \implies 1=r^2, \text{ so } \underline{\underline{r=1}}. \quad (*)$$

$$z=1-x^2-y^2 \implies z=1-r^2$$

rectangular ⟶ cylindrical.

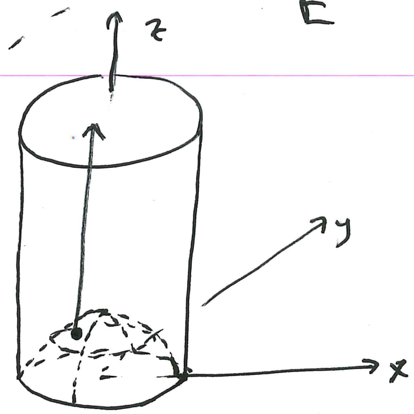
Step 3 : Density function?

distance from z-axis

$$\rho(r,\theta,z) = K\cdot\sqrt{x^2+y^2} = K\cdot r$$

Step 4: Set up integral! (Choose order of integration..)

$$\iiint_E \rho(r,\theta,z)\, dV = \iiint_E (Kr)\cdot dV$$



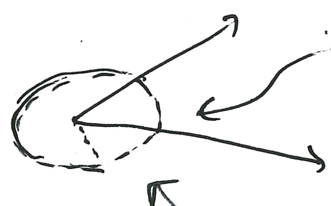
Choosing z first:

start at $z = 1 - r^2$

end at $z = 4$

$$= \underbrace{\iint}_{?} \int_{z=1-r^2}^{z=4} (Kr)\cdot r\, dz\, dr\, d\theta$$

project onto xy-plane (what values (x,y) does "starting at $z=1-r^2$" and ~~ending at~~ "ending at $z=4$" make sense?)



disk of radius 1 $(r=1)$

$$\begin{bmatrix} r=0 \to r=1 \\ \theta=0 \to \theta=2\pi \end{bmatrix}$$

(*) make sure you understand these bounds! (polar integral)

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=1-r^2}^{z=4} (Kr)\cdot r\, dz\, dr\, d\theta$$

Step 5 Compute!

$$\int_0^{2\pi}\int_0^1\int_{1-r^2}^4 Kr^2\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 Kr^2\left(4-(1-r^2)\right)dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (4Kr^2 - Kr^2 + Kr^4)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (3Kr^2 + Kr^4)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[Kr^3 + \frac{Kr^5}{5}\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\left(\left(K+\frac{K}{5}\right)-0\right)d\theta$$

$$= \int_0^{2\pi}\frac{6K}{5}\,d\theta$$

$$= \frac{6K}{5}\cdot\theta\Big]_0^{2\pi}$$

$$= \boxed{\frac{12\pi K}{5}}$$

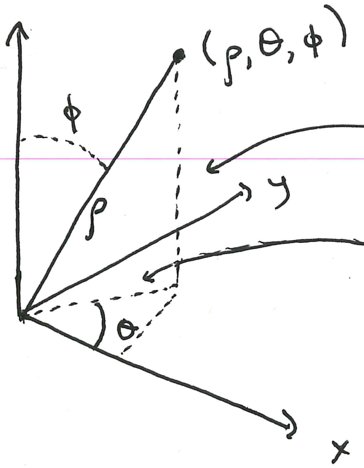
Exercises • Try Example 4 in 15.7 (8th Ed book) / 15.8 (webassign).

• Evaluate the triple integrals by changing to cylindrical coordinates:

① $$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$$

② $$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$$

<u>DEF</u> Spherical Coordinates



$$(*) \quad \boxed{\rho \geq 0, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi}$$

$[z = \rho\cos\phi$

$\underbrace{r = \rho \sin\phi}$ $\longrightarrow$ $\left[\begin{array}{l} x = \rho\sin\phi \cdot \cos\theta \\ y = \rho\sin\phi \cdot \sin\theta \end{array}\right.$

$$\left\{\begin{array}{l} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi \cdot \sin\theta \\ z = \rho\cos\phi \end{array}\right. \qquad \left\{\begin{array}{l} \rho^2 = x^2+y^2+z^2 \\ \left[\text{Don't worry about } \theta, \phi \ \text{(right now)}\right] \end{array}\right.$$

<u>Notice</u> In spherical coordinates, a sphere of radius 1, $x^2+y^2+z^2=1$, has the equation $\rho = 1$.

<u>Exercises</u>

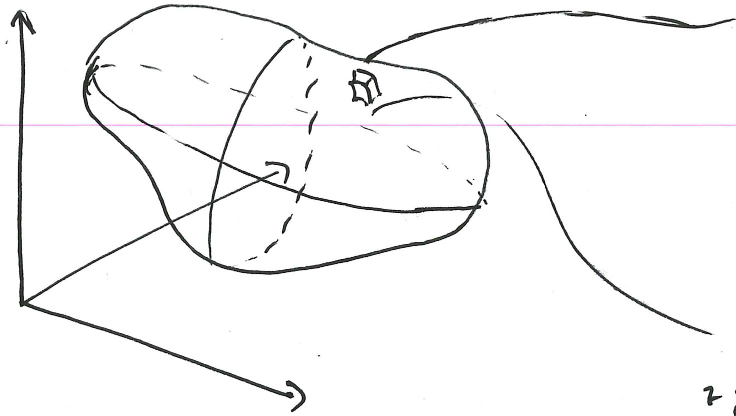
- Convert from spherical to rectangular coordinates:

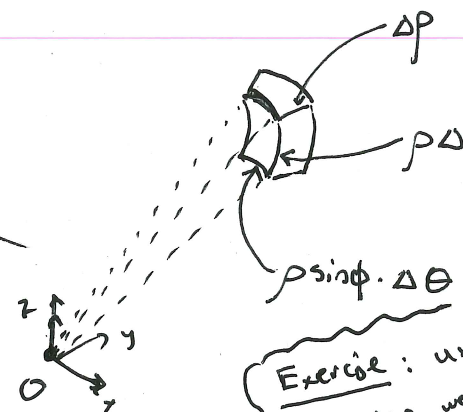
  $(6, \pi/3, \pi/6)$, $(4, -\pi/4, \pi/3)$, $(2, \pi/2, \pi/2)$

- Convert from rectangular to spherical coordinates:

  $(0, -2, 0)$, $(\sqrt{3}, -1, 2\sqrt{3})$, $(1, 0, \sqrt{3})$

Question: Given a volume in 3D space, how do we integrate over this volume in spherical coordinates?





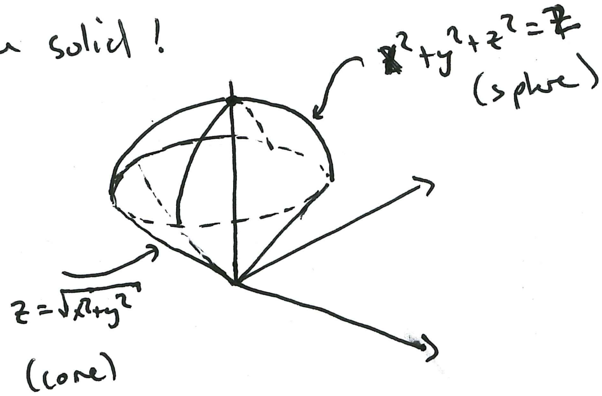
Exercise: use techniques from when we did this with polar integrals to justify "$\rho\Delta\phi$" and "$\rho\sin\phi\,\Delta\theta$" (*)

Then, $\Delta V \approx \rho\sin\phi\cdot\Delta\theta\cdot\rho\,\Delta\phi\cdot\Delta\rho$. In the limit, we get

$$\boxed{dV = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$$

EXAMPLE 2 Use spherical coordinates to find the volume of the solid that lies above $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.

Step 1: Identify the solid!



sphere has center $(0,0,\frac{1}{2})$

Step 2: Write equations in spherical coordinates

sphere
$$\left[\; x^2+y^2+z^2 = z \quad \longrightarrow \quad \rho^2 = \rho\cos\phi \;\text{ OR }\; \rho = \cos\phi \;\right]$$

"rectangular" "spherical"

cone
$$\left[\; z = \sqrt{x^2+y^2} \quad \longrightarrow \quad \rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} \right.$$

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\,(\cos^2\theta+\sin^2\theta)}$$

$$\rho\cos\phi = \rho\sin\phi$$

$$\cos\phi = \sin\phi \,, \quad \underline{0 \le \phi \le \pi}$$

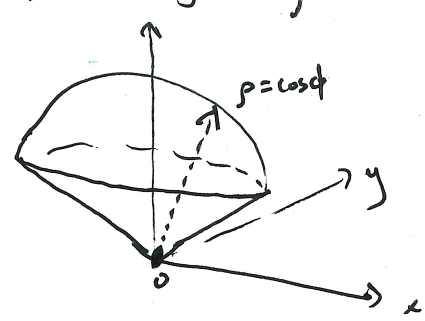
$$\left. \boxed{\phi = \frac{\pi}{4}} \;\right]$$

"rectangular" "spherical"

Step 3 Set up integral! (Choose order of integration.)

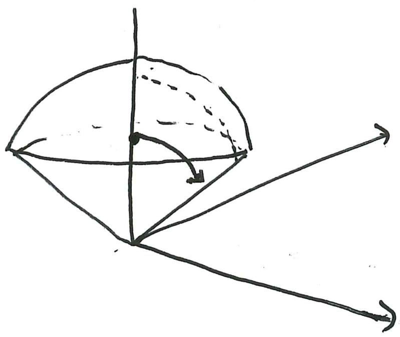
$$\underbrace{\text{Volume} = \iiint_E dV}$$

integrate ρ first.



$$\iiint_E dV = \underbrace{\iint}_{?} \int_{\rho=0}^{\rho=\cos\phi} \rho^2 \sin\phi \, d\rho \, \underline{d\phi\, d\theta}$$

choose ϕ second..



$\phi = 0 \longrightarrow \phi = \frac{\pi}{4}$

integrate θ in a full circle!

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/4} \int_{\rho=0}^{\rho=\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta .$$

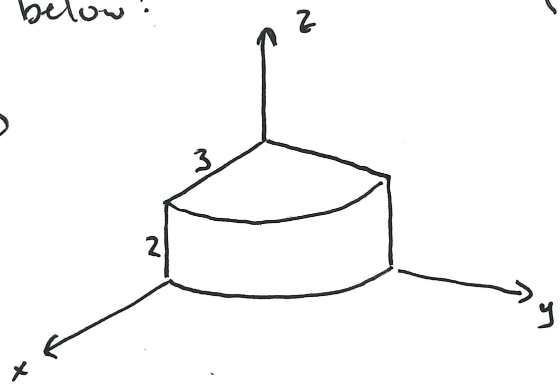
<u>Step 4</u>: Compute!

(<u>Exercise</u> : You should get $\frac{\pi}{8}$.)

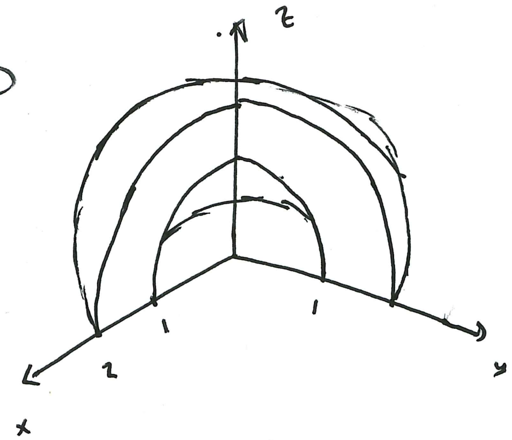
<u>Exercises</u>

- Set up triple integral of $f(x,y,z)$ over the regions below! (use cylindrical or spherical!)

①



②



- Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\; e^{-(x^2+y^2+z^2)}\, dx\, dy\, dz = 2\pi .$$

(The improper integral is defined as the limit of a triple integral over a solid sphere as the radius of the sphere increases indefinitely.)

- Evaluate the integrals by changing to spherical coordinates

① $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\, dz\, dy\, dx$$

② $$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\, dz\, dy\, dx .$$