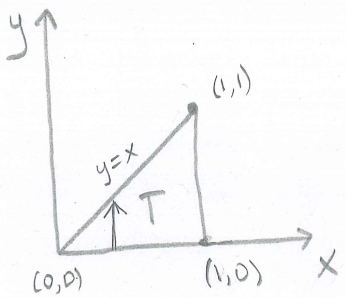
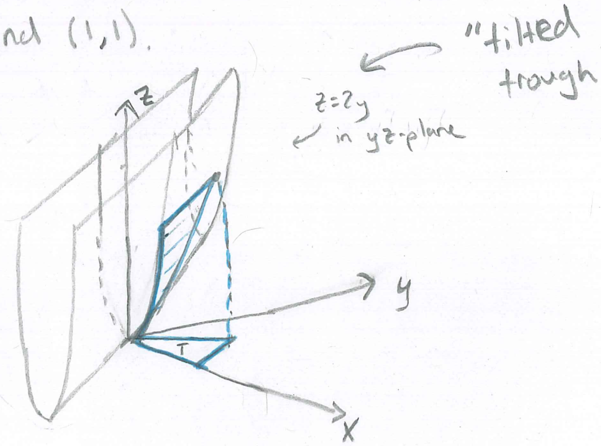
Lecture #5

(15.5 Surface Area continued)

EXAMPLE 1 Find the surface area of the part of the surface: $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0,0)$, $(1,0)$, and $(1,1)$.





Surface Area above T: $SA = \iint_T \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$

$$\frac{\partial z}{\partial x} = 2x \rightsquigarrow \left(\frac{\partial z}{\partial x}\right)^2 = 4x^2$$

$$\frac{\partial z}{\partial y} = 2 \rightsquigarrow \left(\frac{\partial z}{\partial y}\right) = 4$$

So,

$$SA = \iint_T \sqrt{1 + 4x^2 + 4}\, dA$$

(Choose an order to integrate) integrate w.r.t. y first. (try the other way... does it work?)

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x} \sqrt{5+4x^2}\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left[\left(\sqrt{5+4x^2}\right) y \right]_{y=0}^{y=x} dx$$

$$= \int_{x=0}^{x=1} \sqrt{(5+4x^2)} \cdot x \, dx$$

$\downarrow$ u-sub! (this is why we integrated w.r.t. y first!)

$$\begin{cases} u = 5+4x^2 \\ du = 8x\,dx \rightsquigarrow \frac{1}{8}du = x\,dx \end{cases}$$

Bounds: $\begin{cases} x=0 \longrightarrow u=5 \\ x=1 \longrightarrow u=9 \end{cases}$

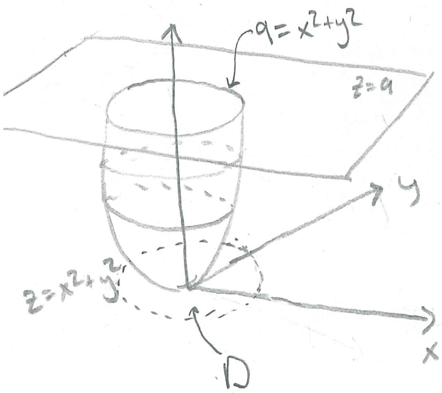
$$= \frac{1}{8} \int_{u=5}^{u=9} \sqrt{u} \, du$$

$$= \frac{1}{8} \left[\frac{2}{3} u^{3/2} \right]_{u=5}^{u=9}$$

$$= \frac{1}{8} \cdot \frac{2}{3} \left(9^{3/2} - 5^{3/2} \right)$$

$$\boxed{= \frac{1}{12} \left(27 - 5\sqrt{5} \right)}.$$

<u>EXAMPLE 2</u> Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies <u>under the plane $z=9$</u>.



<u>Notice</u>! $9 = x^2+y^2 \rightsquigarrow$ this is a disk of radius $\underline{\underline{3}}$.

We can make this our <u>bound</u> for integrating ....

Why?

(We'll call this region D.)

$$SA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = 2y$

$$= \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA$$

use polar! (integrating over a circle and $4x^2+4y^2 = 4(x^2+y^2) = 4r^2$)

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=3} \sqrt{1+4r^2}\, r\,dr\,d\theta$$

u-sub! $\begin{cases} u = 1+4r^2 \\ du = 8r\,dr \Rightarrow \frac{1}{8}du = r\,dr \end{cases}$

$$= \frac{1}{8}\int_0^{2\pi}\left(\int_{r=0}^{r=3} (\sqrt{u})\, du\right) d\theta$$

careful... we didn't change the bounds this time!

$$= \frac{1}{8}\int_0^{2\pi} \left[\frac{2}{3}(u^{3/2})\right]_{r=0}^{r=3} d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \left[\frac{2}{3}(1+4r^2)^{3/2}\right]_0^3 d\theta$$

$$= \frac{1}{8}\cdot\frac{2}{3}\int_0^{2\pi} (37^{3/2} - 1)\, d\theta$$

$$= \frac{1}{12}(37\sqrt{37} - 1)\cdot\int_0^{2\pi} d\theta$$

$$= \frac{1}{12}(37\sqrt{37} - 1)\cdot 2\pi$$

$$\boxed{= \frac{\pi}{6}(37\sqrt{37} - 1)}$$

EXERCISES

① Let $z = ax + by + c$ be a plane. Show that the area on the plane $z = ax + by + c$ above any region D in the xy-plane is $\sqrt{1 + a^2 + b^2} \cdot A(D)$, where $A(D)$ is the area of D.

② Alternatively, pick any region D on the plane $z = ax + by + c$ and let $\pi(D)$ be the projection of that region onto the xy-plane ("think D casts a shadow on the xy-plane", and $\pi(D)$ is the shadow). Show that the area of D is $\sqrt{1 + a^2 + b^2} \cdot A(\pi(D))$.