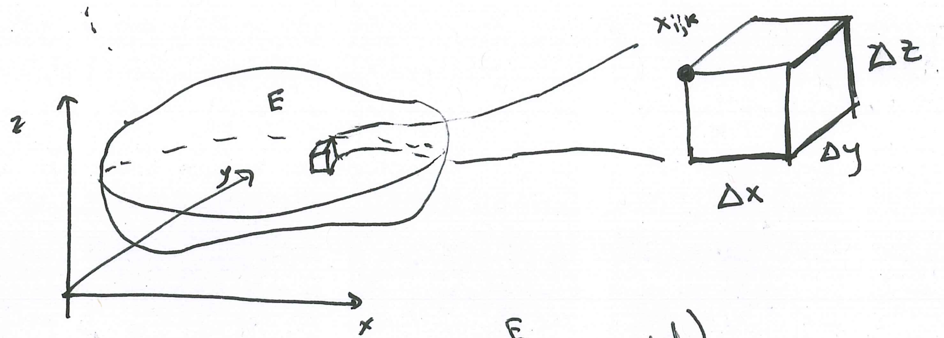
Lecture #5

15.7 (Webassign) / 15.6 (Book) Triple Integrals

(Instructor note: · Determining sets of points
· Mechanics)

DEF

$$\iiint_E f(x,y,z)\, dV := \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}, y_{ijk}, z_{ijk}) \cdot \underbrace{\Delta x \Delta y \Delta z}_{\Delta V}$$



REMARK Provided F is continuous, (region E is bounded) we can integrate with respect to any variable first! In fact, there are 6 different ways to put the integral together!

- dx dy dz
- dx dz dy
- dy dx dz
- dy dz dx
- dz dx dy
- dz dy dx

(Fubini for triple integrals!)

EXAMPLE 1

Evaluate $\iiint_B e^{z}\, dV$ where $B = \{(x,y,z) \mid 0 \le x \le 1,\ -1 \le y \le 3,\ 0 \le z \le 2\}$

$$\iiint_B e^{z}\, dV = \int_0^2 \int_{-1}^3 \int_0^1 e^{z}\, dx\, dy\, dz$$

$$= \int_0^1 e^{z} \left(\int_{-1}^3 \int_0^1 dx\, dy \right) dz$$

$$= \int_0^2 e^{z} \int_{-1}^3 [x]_0^1\, dy\, dz$$

$$= \int_0^2 e^{z} \int_{-1}^3 dy\, dz$$

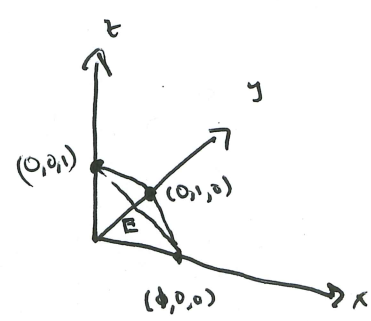
$$= \int_0^2 e^{z} [y]_{-1}^3\, dz$$

$$= 4 \int_0^2 e^{z}\, dz$$

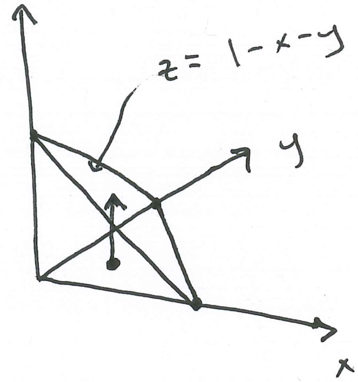
$$= 4\, [e^{z}]_0^2\, dz = \boxed{4e^2 - 4} \overset{\text{OR}}{=} 4(e^2 - 1)$$

EXAMPLE 2

Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$, and $x+y+z=1$.



First: pick a direction to integrate. Lets start with z.

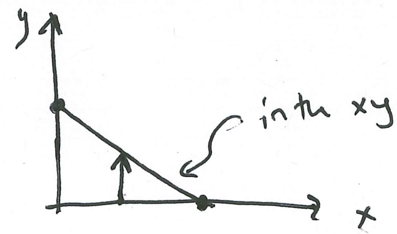


go from $z=0$ to $z=1-x-y$.

$$\underbrace{\iint}\ \underbrace{\int_{z=0}^{z=1-x-y} z\,dz}_{\checkmark}\ \underbrace{d?\,d?}$$

Next, pick the next direction to integrate. Let's do y.

"Think: valid values of y given the boundary on z above."



in the xy-plane, this is $y = 1-x$

$$\int_{x=?}^{x=?} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} z \quad dz\, dy\, dx$$

The last integral will be over valid values of x in the region, i.e. from 0 to 1.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \ dz\, dy\, dx.$$

Exercise Compute this integral! You should get $\frac{1}{24}$.

EXAMPLE 3 Rewrite ~~Express~~ the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, dz\, dy\, dx$ ~~as a triple integral and then rewrite it~~ as an iterated integral in a different order.

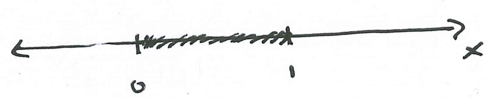
Step 1 figure out the domain!

Two ways: ① Start outside and work in

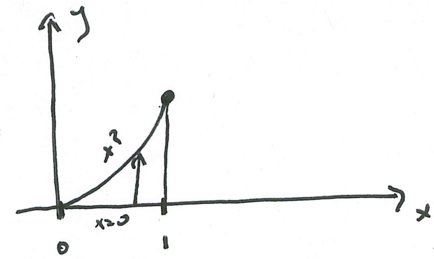
② Start inside and work out

① seems easiest to me:

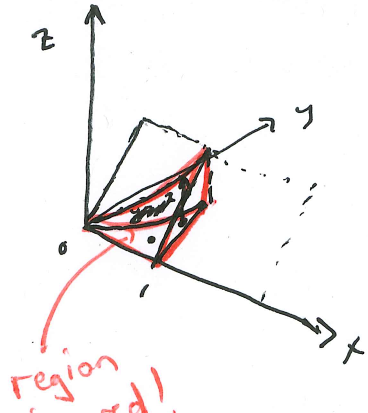
x: must be between 0 and 1



y: must be between 0 and x^2



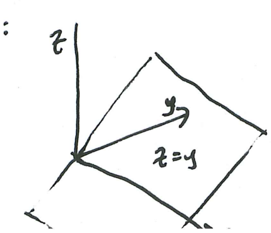
z: must be between $\underbrace{0 \text{ and } y}$



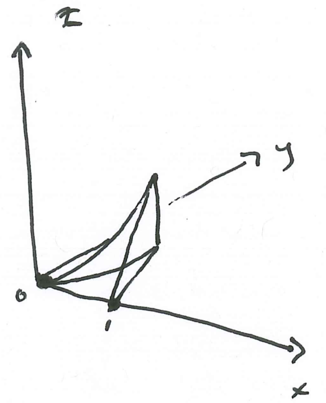
$z=0$ and $z=y$] planes!

$z=0$ is the xy-plane

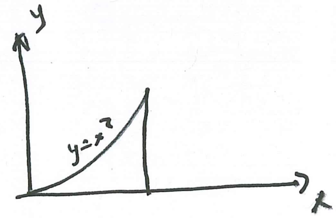
$z=y$:

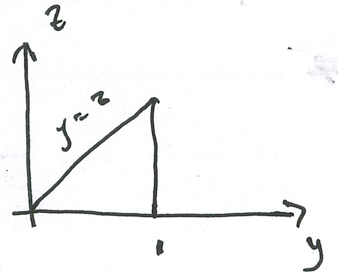


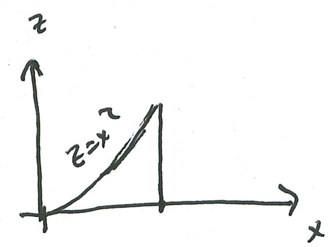
So, our region looks like



Look at projections onto each of the planes : xy-plane
xz-plane
yz-plane





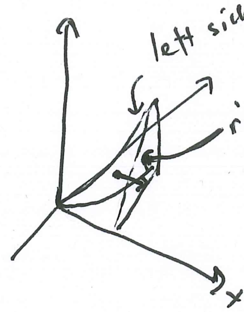


Choose an order of integration:

- $dx\,dz\,dy$

integrating x first:

left side: $y=x^2 \Rightarrow x=\sqrt{y}$

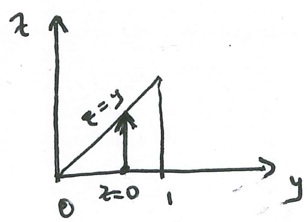


right side $\Rightarrow x=1$

$$\underbrace{\int\int}_{(*)} \underbrace{\int_{x=\sqrt{y}}^{x=1} f(x,y,z)\,dx}_{\checkmark} \;\; \underbrace{dz\,dy}_{(*)}$$

integrating z next: think "x is gone, look at the y-z plane!"

project onto yz-plane!



$$\underbrace{\int}_{*} \int_{z=0}^{z=y} \int_{x=\sqrt{y}}^{x=1} f(x,y,z)\,dx\,dz\,\underbrace{dy}_{*}$$

integrating y last: We see that y is valid from 0 to 1

$$\boxed{\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy.}$$

Exercise Do this for two other orders of integration!

- - - - - - - - -

Exercise • Read through "Section 15.6 Triple Integrals" in the book, specifically "Applications of Triple Integrals" starting on p. 1074.

• Work through Example 6.

REMARK . We can think of a triple integral as a "4-dim" volume.

Or as a volume:

$$\underbrace{\iiint_E dV}_{\text{"sum up little pieces of volume"}} = \iiint_E dx\,dy\,dz$$

. This is analogous to thinking of double integrals as areas of regions.

. In other words, EXAMPLE 2 could have been phrased

(*) "Find the volume of the solid, E, bounded by the four planes $x=0$, $y=0$, $z=0$, and $x+y+z=1$."

Instead of integrating $\iiint_E z\,dV$, we would have integrated $\iiint_E dV$.

<u>NOTE</u> On a test, I won't give equations for moment of inertia or center of mass as described in section 15.4 (HW 15.5). However, if I ask you to compute the moment of inertia or center of mass of a 3-dimensional object, which is the subject of Example 6 in the exercise above, I will give these equations to you. However, you do have one homework problem on this topic, so you will need to do the exercise above!

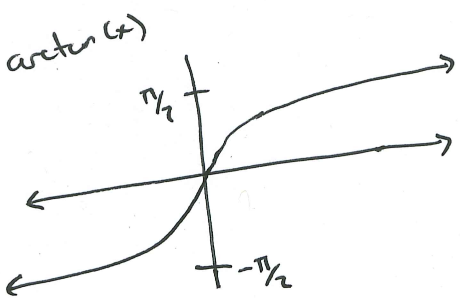
<u>More Exercises</u> : #34 in the book! (15.6).

Improper Double Integrals

<u>Ex</u> $\underbrace{\int_0^\infty \int_0^\infty}_{\text{First Quadrant!}} \frac{1}{(1+x^2)(1+y^2)} \, dx\,dy = \left(\int_0^\infty \left(\lim_{b\to\infty} \int_0^b \frac{1}{(1+x^2)(1+y^2)} \, dx \right) dy \right)$

$$= \int_0^\infty \lim_{b\to\infty} \Big[\arctan(x) \Big]_0^b \cdot \frac{1}{1+y^2} \, dy$$

$$= \int_0^\infty \left(\lim_{b\to\infty} \arctan(b) \right) \cdot \frac{1}{1+y^2} \, dy$$



$$= \int_0^\infty \left(\frac{\pi}{2}\right) \cdot \frac{1}{1+y^2} \, dy$$

$$= \frac{\pi}{2} \cdot \lim_{b\to\infty} \int_0^b \frac{1}{1+y^2} \, dy$$

$$= \frac{\pi}{2} \left(\lim_{b\to\infty} \arctan(b) \right) = \frac{\pi}{2} \cdot \frac{\pi}{2} = \boxed{\frac{\pi^2}{4}}$$

Ex $\displaystyle\int_0^\infty\int_0^\infty \frac{1}{1+x^2+y^2}\,dx\,dy = \iint_{Q_1} \frac{1}{1+x^2+y^2}\,dA$

Q_1 = First Quadrant

polar!

$$= \int_0^{\pi/2}\int_0^\infty \frac{1}{1+r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \lim_{b\to\infty}\int_0^b \frac{1}{1+r^2}\,r\,dr\,d\theta$$

u-sub!

$$= \int_0^{\pi/2} \lim_{b\to\infty} \frac{1}{2}\int_0^b \frac{1}{u}\,du\,d\theta$$

$$= \int_0^{\pi/2} \left(\lim_{b\to\infty} \frac{1}{2}\left[\ln(u)\right]_0^b\right) d\theta$$

$$= \int_0^{\pi/2} \underbrace{\lim_{b\to\infty} \frac{1}{2}\left(\ln(b) - 1\right)}\,d\theta$$

Limit is ∞,

So this does <u>not</u> converge!

Ex ~~ercise~~ Write the bounds $\displaystyle\int_{-\infty}^\infty\int_{-\infty}^\infty f(x,y)\,dx\,dy$ in polar coordinates.

Solution: $\displaystyle\int_0^{2\pi}\int_0^\infty f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

$$= \int_0^{2\pi} \lim_{b\to\infty}\int_0^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Exercise See book problem #40 in 8^{th} ed. Section 15.3

(15.4 in Webassign ~ polar coordinates.)