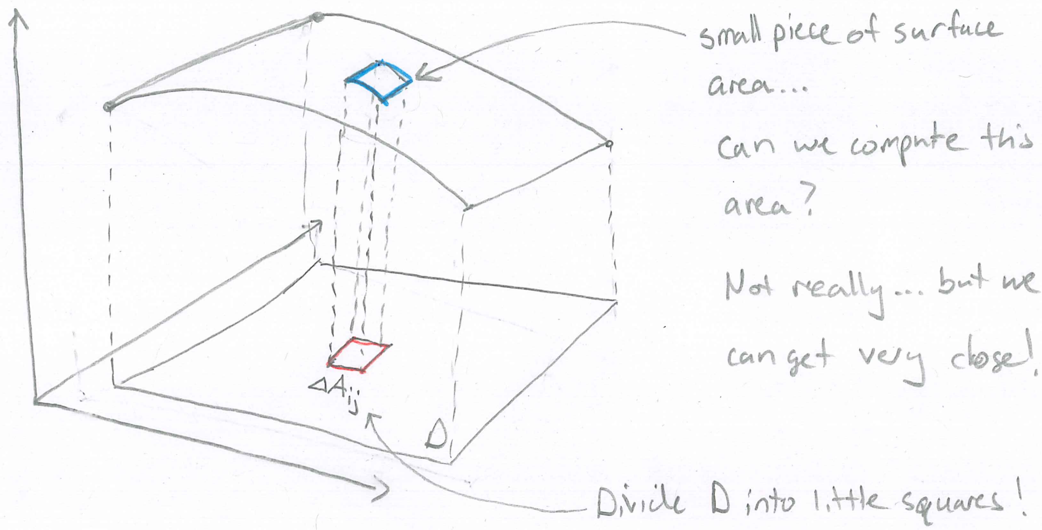
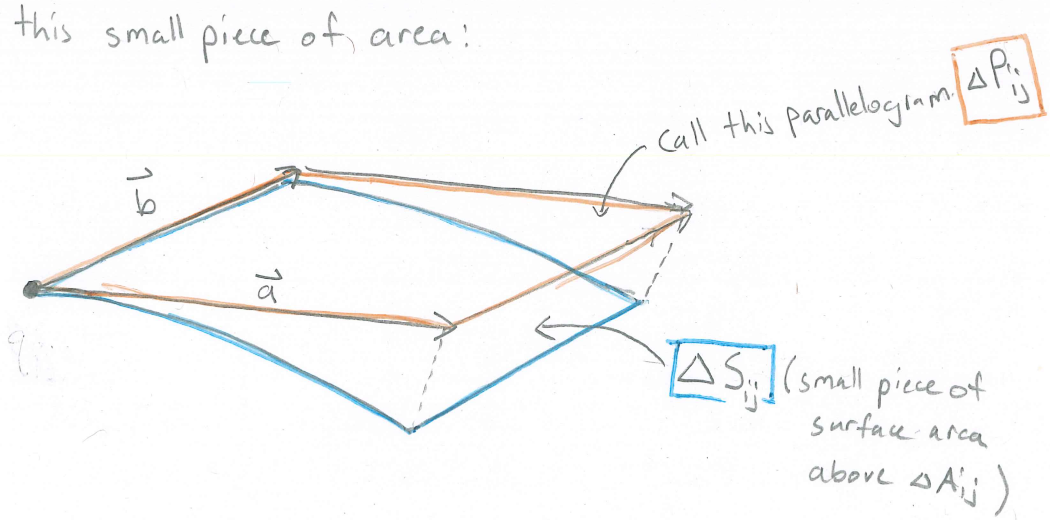
15.5 Surface Area

REMARK We will revisit surface area in Chapter 16, defining it with vector-valued functions.

GOAL Develop a formula that will enable us to compute the area of a surface (for now, this means graph) above a region in the xy-plane.

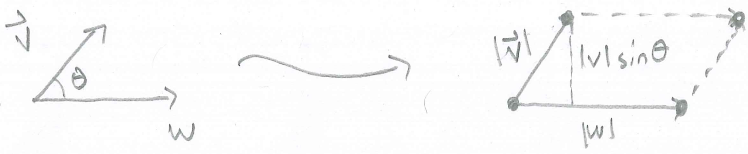


Look closely at this small piece of area:



We can estimate the area of ΔS_{ij} by using the area of ΔP_{ij}. To compute the area of ΔP_{ij}, we need to understand what $\vec{a}$ and $\vec{b}$ are ... and how to use $\vec{a}$ and $\vec{b}$ to compute the area!

REM Area of a parallelogram



$$\text{Area} = |\vec{w}|\,|\vec{v}|\cdot\sin\theta$$

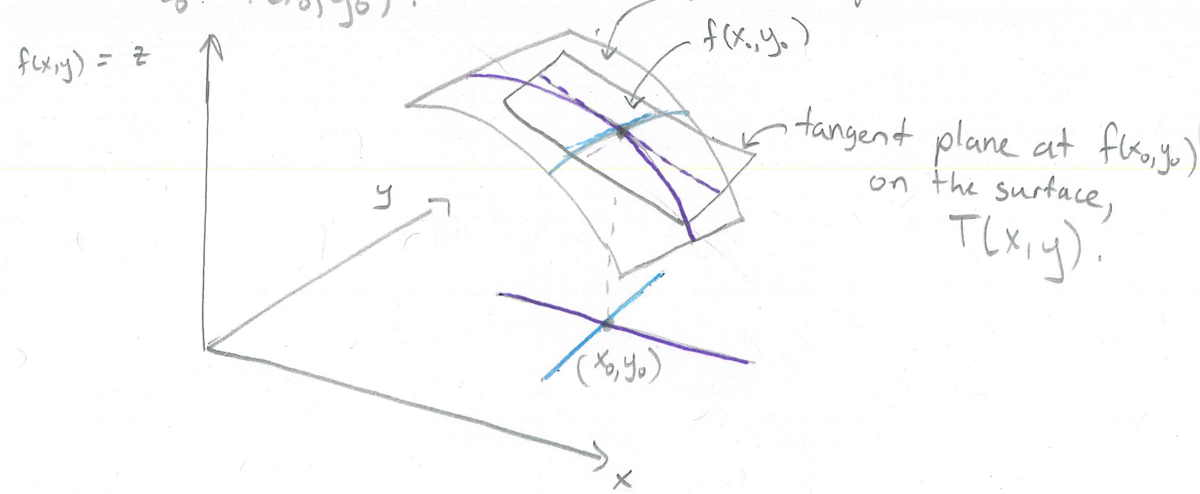
$$= |\vec{w}\times\vec{v}| \quad \left(= |\vec{v}\times\vec{w}|, \text{ but } \vec{w}\times\vec{v} = -(\vec{v}\times\vec{w})\right)$$

So, we can estimate ΔP_{ij} with $|\vec{a}\times\vec{b}|$. But what are $\vec{a}$ and $\vec{b}$?

REM Tangent planes at a point $f(x_0,y_0)$ on a surface (see 14.4 in text!).

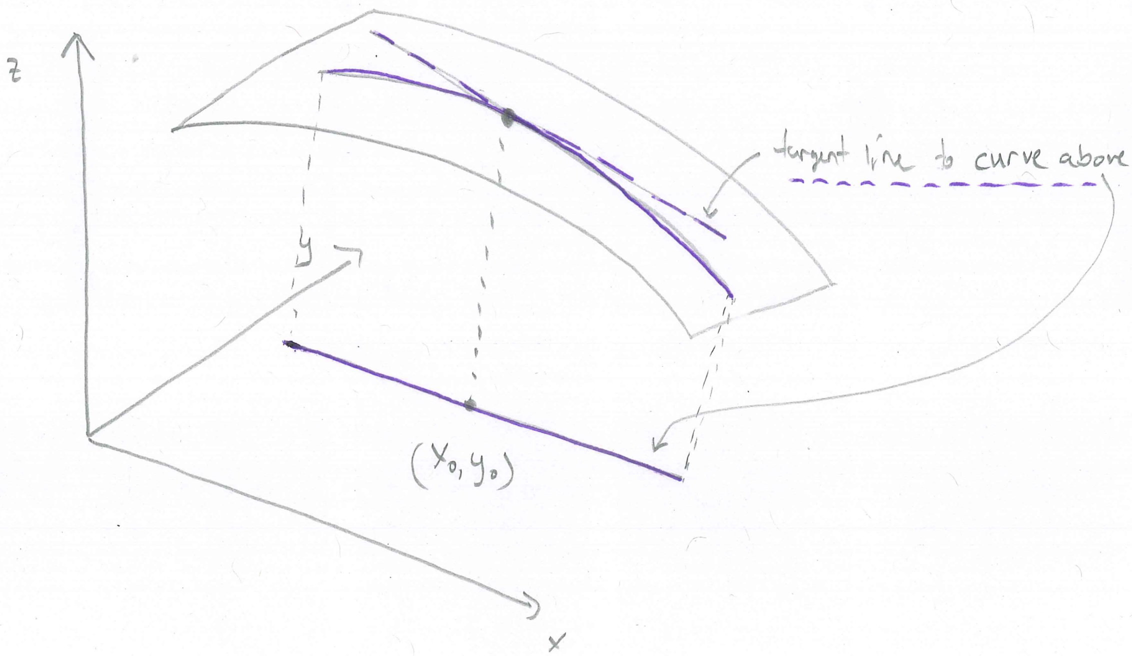
$$T(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

where $z_0 = f(x_0,y_0)$.



Now, look at $\boxed{\text{a line passing through } (x_0,y_0)}$, moving only in the x-direction. There is a corresponding curve on the surface above.

Now, if you find the line tangent to that curve at $f(x_0, y_0)$, it must live in the tangent plane!



QUESTION What is the slope of this line?

It's the partial derivative of f in the x direction evaluated at (x_0, y_0), i.e.

$$\boxed{f_x(x_0, y_0)}$$

We can do the same in the y-direction, and we see a line with slope

$$\boxed{f_y(x_0, y_0)}.$$

Now, return to the formula for the tangent plane:

$$T(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

so

$$\underbrace{(T(x,y) - z_0)}_{\text{think } \Updownarrow \; \Delta z} = f_x(x_0,y_0)\underbrace{(x-x_0)}_{\Updownarrow \; \Delta x} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{\Updownarrow \; \Delta y}.$$

$$\Delta z = f_x(x_0,y_0)\cdot\Delta x + f_y(x_0,y_0)\cdot\Delta y.$$

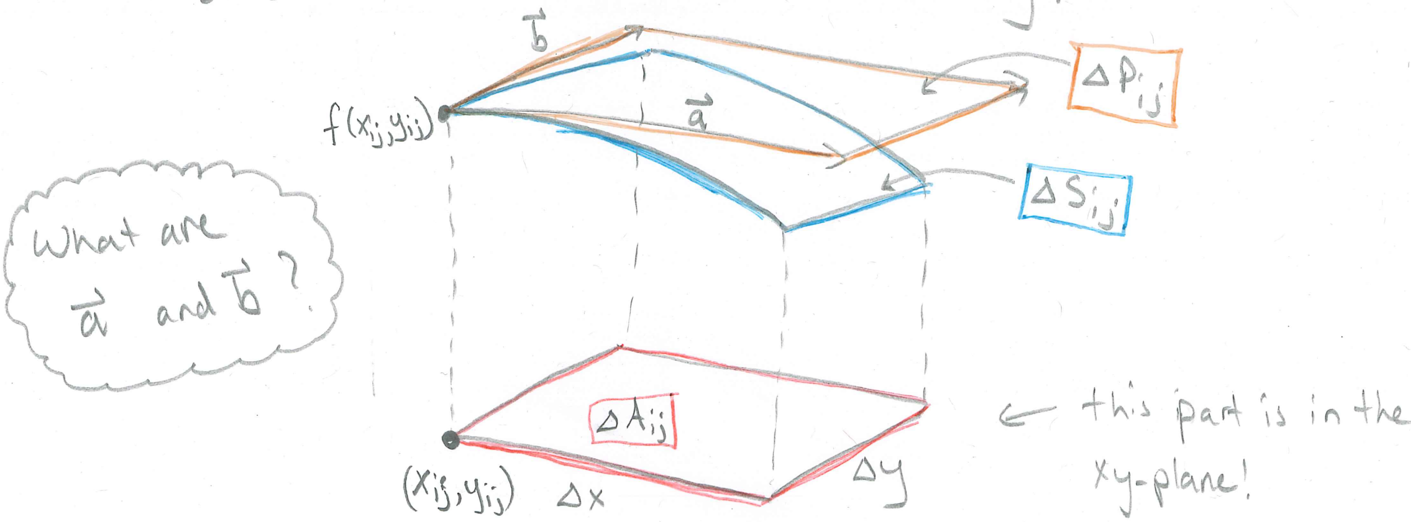
If we move in the tangent plane <u>only in the x-direction</u>, then Δy is 0 and we see

$$\boxed{\Delta z = f_x(x_0,y_0)\cdot\Delta x} \quad \text{(equation of the tangent line!)}$$

Similarly, if we move <u>only in the y-direction</u>, then Δx is 0 and

$$\boxed{\Delta z = f_y(x_0,y_0)\cdot\Delta y}.$$

Now, let's return to our original question, and lets make sure to ask it in a well-defined way:



Let's do $\vec{a}$ first. Along $\vec{a}$, there is no change in y, but there is a change in x, namely Δx. So

$$\vec{a} = \langle \Delta x, 0, (?) \rangle$$

How much does z change?

$$\Delta z = f_x(x_{ij}, y_{ij}) \cdot \Delta x \quad \left(+ f_y(x_{ij}, y_{ij}) \cdot \overset{\Delta y}{\overset{\downarrow}{0}} \right)$$

So, we can conclude

$$\vec{a} = \langle \Delta x, 0, \Delta z \rangle$$

$$\boxed{\vec{a} = \langle \Delta x, 0, f_x(x_{ij}, y_{ij}) \cdot \Delta x \rangle.}$$

Similarly for $\vec{b}$, we see

$$\vec{b} = \langle 0, \Delta y, \Delta z \rangle$$

where

$$\Delta z = f_y(x_{ij}, y_{ij}) \cdot \Delta y \quad \left(+ f_x(x_{ij}, y_{ij}) \cdot \overset{\Delta x}{\overset{\downarrow}{0}} \right),$$

so

$$\boxed{\vec{b} = \langle 0, \Delta y, f_y(x_{ij}, y_{ij}) \cdot \Delta y \rangle}$$

Now we can compute Area (ΔP_{ij}):

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \Delta x & 0 & f_x(x_{ij}, y_{ij})\,\Delta x \\ 0 & \Delta y & f_y(x_{ij}, y_{ij})\,\Delta y \end{vmatrix}$$

$$= -f_x(x_{ij}, y_{ij})\cdot\overbrace{\boxed{\Delta x \Delta y}}^{\Delta A_{ij}}\,\hat{\imath} + -f_y(x_{ij}, y_{ij})\cdot\overbrace{\boxed{\Delta x \Delta y}}^{\Delta A_{ij}}\,\hat{\jmath} + \overbrace{\boxed{\Delta x \Delta y}}^{\Delta A_{ij}}\,\hat{k}$$

And:

$$|\vec{a} \times \vec{b}| = \sqrt{f_x^2(x_{ij}, y_{ij})\,(\Delta A_{ij})^2 + f_y^2(x_{ij}, y_{ij})\cdot(\Delta A_{ij})^2 + (\Delta A_{ij})^2}$$

$$= \left(\sqrt{f_x^2(x_{ij}, y_{ij}) + f_y^2(x_{ij}, y_{ij}) + 1}\right)\cdot \Delta A_{ij}$$

$$= \sqrt{1 + f_x^2(x_{ij}, y_{ij}) + f_y^2(x_{ij}, y_{ij})}\cdot \Delta A_{ij}$$

So for each $\boxed{\Delta A_{ij}}$, we can approximate the surface area $\boxed{\Delta S_{ij}}$ using $\boxed{\Delta P_{ij}}$. We just have to sum over all $\boxed{\Delta A_{ij}}$'s (just add them all up!)

Area (Surface above region D)

$$= \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \overbrace{\sqrt{1 + f_x^2(x_{ij}, y_{ij}) + f_y^2(x_{ij}, y_{ij})}\cdot \Delta A_{ij}}^{\text{Area}(\Delta P_{ij})}$$

$$= \iint_D \sqrt{1 + f_x^2(x,y) + f_y^2(x,y)}\, dA$$

Let S be the "surface above region D", and let $A(S)$ be the area of S. We can capture the equation for surface area in a variety of notations, so we will list a few below.

EQUATION Surface Area

$$A(S) = \iint_D \sqrt{1 + f_x^2(x,y) + f_y^2(x,y)}\, dA$$

or

$$= \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

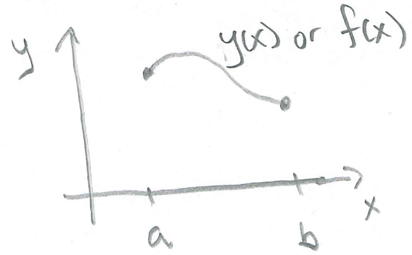
← Alternative notation for partial derivative of $f(x,y)$

or

$$= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

← Thinking of $f(x,y) = z$ (so really, $z(x,y)$).

REMARK Compare this to the formula for arc length you may have seen in M125:



$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

or

$$= \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$