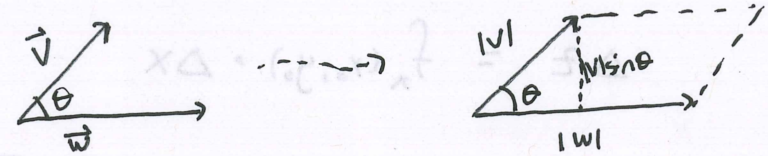
Lecture #4

15.6 (Webassign) / 15.5 (8th ed Book) Surface Area.

rem: $|\vec{v} \times \vec{w}| = |v|\cdot|w|\sin\theta$ = Area of parallelogram

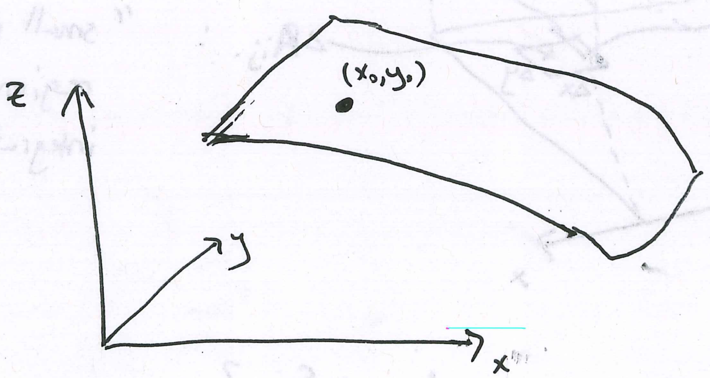
notice:



$|w|\cdot|v|\sin\theta$ = Area of parallelogram!

rem Tangent planes at a point (x_0, y_0) on a surface (see 14.4 in 8th ed!)

$$f(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(y-y_0) + \underbrace{z_0}_{= f(x_0,y_0)}$$



Notice, the "line" in the direction of the x-axis has no change in x, i.e. $x = x_0$ along this line. Plugging in, we see

$$\underbrace{f(x_0,y)}_{z} = f_y^{(x_0,y_0)}(y-y_0) + \underbrace{f(x_0,y_0)}_{z_0}$$

$$z - z_0 = f_y(x_0,y_0)\cdot(y-y_0) \quad \leftarrow \text{equation of a line with slope}$$

$$\text{Think} \quad \Delta z = f_y(x_0,y_0)\cdot\Delta x$$

"change in z" = (partial w.r.t. y) · "change in x"

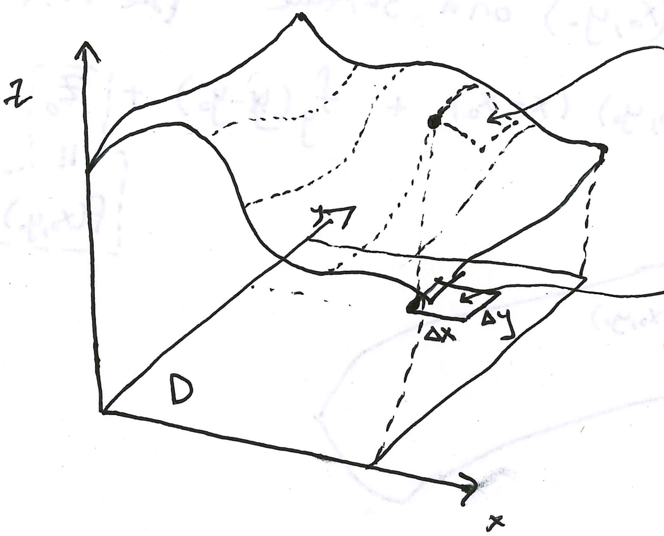
Similarly, the "line" in the direction of the y-axis has no change in y, i.e. $y = y_0$ along this line. So again, we have

Exercise: Derive this

$$z - z_0 = f_x(x_0, y_0) \cdot (x - x_0)$$

$$\Delta z = f_x(x_0, y_0) \cdot \Delta x$$

To compute Surface Area ...



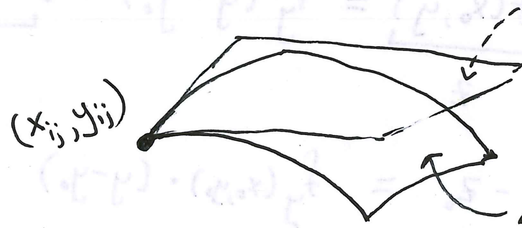
ΔS_{ij} : "small piece of surface area" directly above

ΔA_{ij} : "small piece of the region D we are integrating over"

How do we compute the area of ΔS_{ij}?

We estimate it with a tangent approximation.

Zoom in:



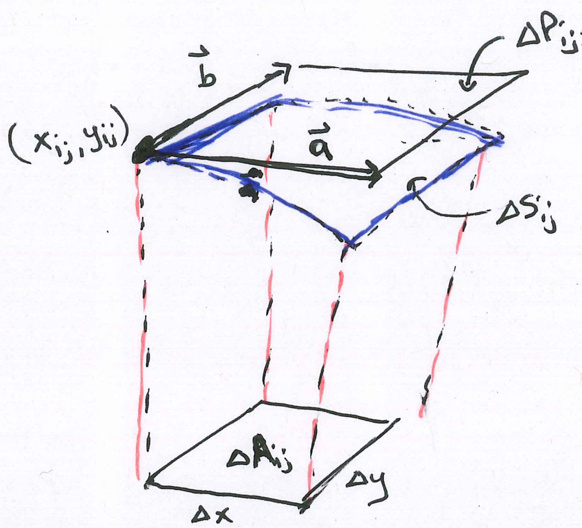
ΔP_{ij} : small parallelogram on the tangent plane!

ΔS_{ij}

REMARK The area of ΔP_{ij} will approach the area of ΔS_{ij} as we let our "mesh" get finer and finer (i.e. as we let ΔA_{ij} get smaller and smaller.)

To compute the area of ΔP_{ij}:

(Notice, ΔP_{ij} could be more tilted!)



$$\text{Area}(\Delta P_{ij}) = |\vec{a} \times \vec{b}|.$$

What are $\vec{a}$ and $\vec{b}$?

- Along $\vec{a}$, there is no change in y, so

$$\langle \Delta x, 0, ? \rangle$$

"Δz given Δx and no change in y"

to find Δz, rem tangent plane approximation!

$$\Delta z = f_x(x_{ij}, y_{ij}) \cdot \Delta x$$

- Along $\vec{b}$, there is no change in x, so

$$\langle 0, \Delta y, ? \rangle$$

"Δz given a change in y and no change in x"

to find Δz, same as above!

$$\Delta z = f_y(x_{ij}, y_{ij}) \cdot \Delta y$$

So

$$\vec{a} = \langle \Delta x, 0, f_x(x_{ij}, y_{ij}) \cdot \Delta x \rangle$$

$$\vec{b} = \langle 0, \Delta y, f_y(x_{ij}, y_{ij}) \cdot \Delta y \rangle$$

Then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \Delta x & 0 & f_x(x_{ij}, y_{ij})\Delta x \\ 0 & \Delta y & f_y(x_{ij}, y_{ij})\Delta y \end{vmatrix}$

$$= -f_x(x_{ij}, y_{ij}) \underbrace{\Delta x \Delta y}_{\Delta A} \hat{\imath} + -f_y(x_{ij}, y_{ij}) \underbrace{\Delta x \Delta y}_{\Delta A} \hat{\jmath} + \underbrace{\Delta x \Delta y}_{\Delta A} \hat{k}$$

And $|\vec{a} \times \vec{b}| = \sqrt{f_x^2(x_{ij}, y_{ij})(\Delta A)^2 + f_y^2(x_{ij}, y_{ij}) \cdot (\Delta A)^2 + (\Delta A)^2}$

$$= \sqrt{1 + f_x(x_{ij}, y_{ij}) + f_y(x_{ij}, y_{ij})} \cdot \Delta A.$$

Then, for each i and each j, we want to approximate the surface area with a Riemann Sum:

$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{1 + f_x^2(x_{ij}, y_{ij}) + f_y^2(x_{ij}, y_{ij})} \cdot \Delta A.$$

taking the limit!

(*)

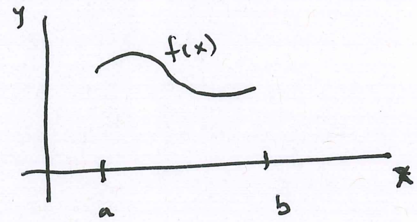
$$= \iint_D \sqrt{1 + f_x^2(x,y) + f_y^2(x,y)} \, dA$$

(*)

$$= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA.$$

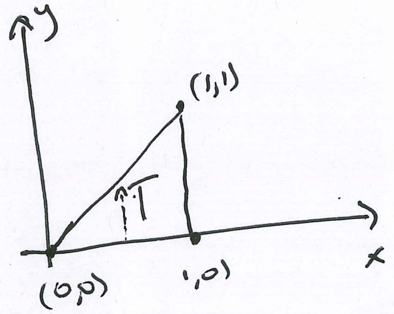
alternative notation.

REMARK Very similar to the formula for arc length you may have seen in m125 :



$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

EXAMPLE 1 Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0,0)$, $(1,0)$, and $(1,1)$.



$$\text{Surface Area} = \iint_T \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$\frac{\partial z}{\partial x} = 2x \rightsquigarrow \left(\frac{\partial z}{\partial x}\right)^2 = 4x^2$$

$$\frac{\partial z}{\partial y} = 2 \rightsquigarrow \left(\frac{\partial z}{\partial y}\right)^2 = 4$$

$$= \iint_T \sqrt{1 + 4x^2 + 4}\, dA$$

integrate w.r.t. y first!

$$= \int^{x=1} \int^{y=x} \sqrt{5 + 4x^2}\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left[\left(\sqrt{5+4x^2}\right) y \right]_0^x \, dx$$

$$= \int_{x=0}^{x=1} \left(\sqrt{5+4x^2}\right) \cdot x \, dx$$

u-sub! (why we integrated w.r.t. y first!)

$u = 5+4x^2$

$du = 8x \, dx$

$x=0 \Rightarrow u=5$

$x=1 \Rightarrow u=9$

$$= \frac{1}{8} \int_{x=0}^{x=1} \sqrt{u} \, du$$

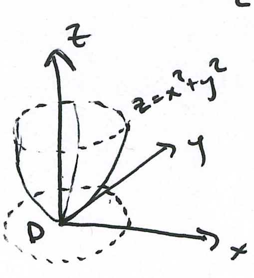
$$= \frac{1}{8} \left[\frac{2}{3} \cdot u^{3/2} \right]_{u=5}^{u=9}$$

$$= \frac{1}{8} \cdot \frac{2}{3} \left(9^{3/2} - 5^{3/2} \right)$$

$$\boxed{= \frac{1}{12} \left(27 - 5\sqrt{5} \right)}$$

EXAMPLE 2 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z=9$.

$\hookrightarrow 9 = x^2 + y^2$



$$SA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

D is a ~~circle~~ disk with radius 3!

$$\frac{\partial z}{\partial x} = 2x \quad , \quad \frac{\partial z}{\partial y} = 2y$$

$$= \iint_D \sqrt{1 + 4x^2 + 4y^2} \, dA$$

use polar!

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=3} \sqrt{1+4(x^2+y^2)}\; r\,dr\,d\theta$$

u-sub... !

$$= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\; r\,dr\,d\theta$$

$u = 1+4r^2$

$du = 8r\,dr$

$r=0 \Rightarrow u=1$

$r=3 \Rightarrow u=37\ldots$

$$= \frac{1}{8}\int_0^{2\pi}\int_{r=0}^{r=3} \sqrt{u}\,du\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}\left[\frac{2}{3}u^{3/2}\right]_{r=0}^{r=3} d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}\left[\frac{2}{3}(1+4r^2)^{3/2}\right]_{r=0}^{r=3} d\theta$$

$$= \frac{1}{8}\cdot\frac{2}{3}\int_0^{2\pi}(37^{3/2}-1)\,d\theta$$

$$= \frac{1}{12}(37\sqrt{37}-1)\int_0^{2\pi} d\theta$$

$$= \frac{2\pi}{12}(37\sqrt{37}-1) = \boxed{\frac{\pi}{6}(37\sqrt{37}-1)}$$