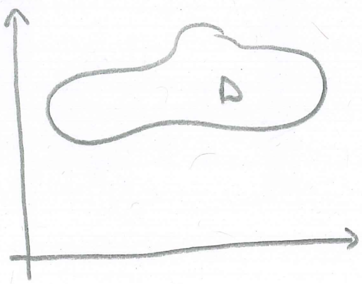
15.4 Applications of Double Integrals

Say we have a region in the plane that is representative of a (flat) object:

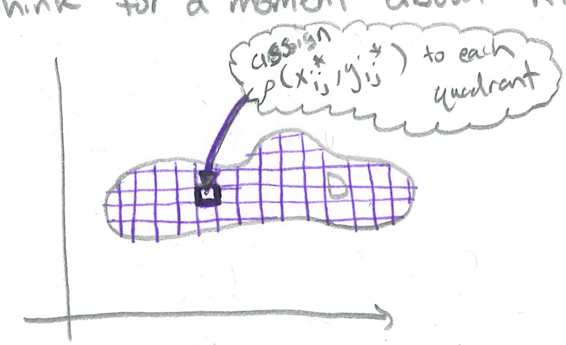


- We could compute the area of this region, $\iint_D dA$, giving us something in "units squared." Assume for a moment that our units are meters, so m^2 is the unit for the area.
- If we had a "density" function, $\rho(x,y)$, which gives us the density at each point in D, we could compute the mass.

<u>RECALL</u>: $$\text{Area} \cdot \text{density} = \text{mass}$$
$$\left(m^2 \cdot kg/m^2 = kg \right)$$

<u>NOTE</u> In the above formula, density may be referred to as an "area density".

- Now, think for a moment about Riemann Sums:



$$m = \lim_{k,\ell \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \underbrace{\rho(x^*_{ij}, y^*_{ij})}_{\text{density}} \cdot \underbrace{\Delta x \Delta y}_{\text{area}}$$

• Taking the limit, we get a new interpretation of an integral:

$$\text{mass} = \boxed{m = \iint_D \rho(x,y)\,dA}$$

QUESTION Did it really matter (in the construction of this integral) what the units were? Sure.... but not really. Sure because we started with area, but no because we didn't end up with area...

What we did: $\text{area} \cdot \underbrace{\left(\frac{\text{mass}}{\text{area}}\right)}_{\text{density!}} = \text{total mass}$

what we could do: $\text{area} \cdot \left(\frac{\text{ANY UNIT!}}{\text{area}}\right) = \text{total ANY UNIT!}$

EXAMPLE Say we had the same region D as above, but instead of an area-density function $\rho(x,y)$, we were given a charge density $\sigma(x,y)$ (units of charge, Coulombs, per unit area).

Then, we could compute the TOTAL CHARGE, Q.

$$\boxed{Q = \iint_D \sigma(x,y)\,dA}$$

QUESTION Given any region D with some area-density function $\rho(x,y)$, we can compute the total mass. Can we find the center of mass? In other words, where could we put our finger to balance the flat plate D:

We can compute this point! However, we will need some definitions first.

DEF Moment about the x-axis

$$\boxed{M_x = \iint_D y\,\rho(x,y)\,dA}$$

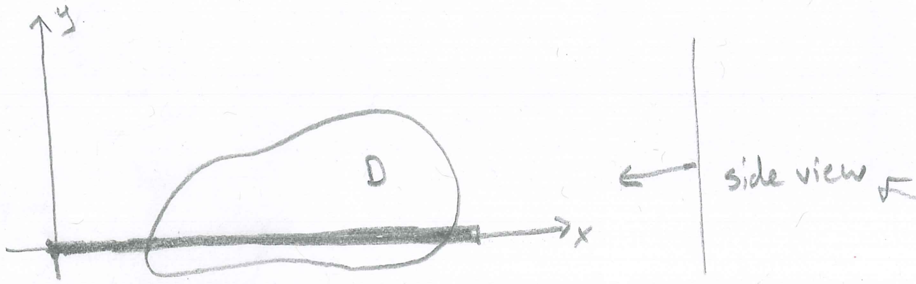
DEF Moment about the y-axis

$$\boxed{M_y = \iint_D x\,\rho(x,y)\,dA}$$

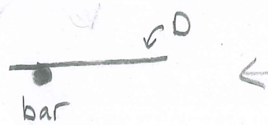
QUESTION What are these moments? Is there some intuition underlying these objects?

INTERPRETATION 1

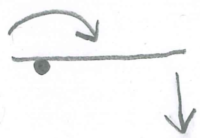
Consider M_x, moment about the x-axis. Imagine a bar beneath the region D along the x-axis:



How "unbalanced" is D on this bar. Consider a side view.



Notice that if more mass is on one side of the bar, there is an immediate "tendency" for D to fall in that direction. In the moment the fall begins, we can think of this as a rotation about the bar, or a rotation about the x-axis.



M_x is giving you some indication as to how quickly D will fall over one side or the other of the bar (or x-axis).

QUESTION Why does M_x have a "y" in the integral?

Because the further the mass is from the x-axis (which is the y-coordinate!) the faster it wants to rotate about the x-axis!

Then, we can interpret M_y the same way, except instead of a "rotation" about the x-axis, it is measuring some tendancy to "rotate" about the y-axis.

INTERPRETATION 2

Consider the Riemann sum whose limit is M_x:

$$\sum y^* \cdot \rho(x^*, y^*) \cdot \Delta x \Delta y \longrightarrow \iint_D y\rho(x,y)\, dA$$

REM $\iint_D \rho(x,y)\, dA = \text{mass} = m$

Consider dividing this Riemann sum by the mass m:

$$\frac{1}{m}\left(\sum y^* \cdot \rho(x^*, y^*)\, \Delta x \Delta y\right).$$

What we have written is a "weighted average", i.e. the average weighted by the y-coordinate.

We can interpret M_y similarly.

Now we can define the center of mass:

DEF center of mass, $(\bar{x}, \bar{y})$

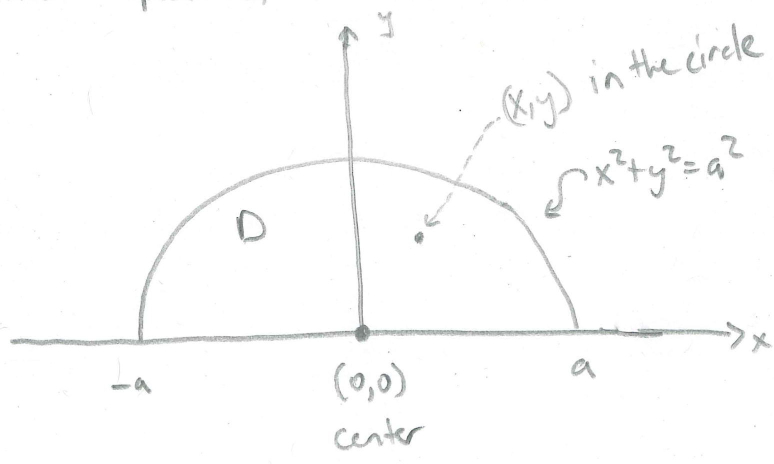
$$\boxed{\begin{aligned} \bar{x} &= \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x,y)\,dA \\ \bar{y} &= \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x,y)\,dA, \end{aligned}}$$

where m is the mass.

REMARK Notice that we are scaling the moment by mass!

EXAMPLE Consider the upper half of the circle $x^2+y^2=a^2$. Assume the density at any point in the half circle is proportional to the distance from the center of the circle. Find the center of mass.

STEP 1: Draw a picture!



STEP 2: Interpret the sentence about the density.

- Distance to the center for any (x,y) in our region:

$$d(x,y) = \sqrt{x^2+y^2}$$

- Density is proportional to distance, so our density function is

$$\rho(x,y) = K \cdot d = K\sqrt{x^2+y^2}$$

STEP 3: Compute the mass.

$$m = \iint_D \rho(x,y)\,dA = \iint_D K\sqrt{x^2+y^2}\,dA$$

use polar!

$r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2}$

$$= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a} (Kr)\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{a} Kr^2\,dr\,d\theta$$

$$= \int_0^{\pi} \left[\frac{K}{3} r^3\right]_0^a d\theta$$

$$= \frac{Ka^3}{3}\int_0^{\pi} d\theta$$

$$= \frac{Ka^3}{3}\cdot\left[\theta\right]_0^{\pi}$$

$$= \boxed{\frac{K\pi a^3}{3}}$$

STEP 4: Compute $\bar{x}$.

(Exercise! This is 0. Can you go back to the statement of the problem, and see why $\bar{x}$ should be 0?)

STEP 5: Compute $\bar{y}$.

polar! $y = r\sin\theta$, $\rho(x,y)$ as before

$$\bar{y} = \frac{1}{m}\iint_D y \cdot \rho(x,y)\, dA = \frac{3}{K\pi a^3}\int_0^{\pi}\int_0^{a} (r\sin\theta)\cdot(Kr)\cdot \overbrace{r\,dr\,d\theta}^{dA}$$

$$= \frac{3}{\pi a^3}\int_0^{\pi}\int_0^{a} r^3 \sin\theta \, dr\, d\theta$$

$$= \frac{3}{\pi a^3}\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^a \cdot \sin\theta \, d\theta$$

$$= \frac{3}{\pi a^3}\int_0^{\pi} \frac{a^4}{4} \sin\theta \, d\theta$$

$$= \frac{3a}{4\pi}\int_0^{\pi} \sin\theta \, d\theta$$

$$= \frac{3a}{4\pi}\left[-\cos\theta\right]_0^{\pi}$$

$$= \frac{3a}{4\pi}\left[-(-1) - (-1)\right]$$

$$= \frac{3a}{4\pi}\cdot 2$$

$$= \frac{3a}{2\pi}$$

So, we can conclude the center of mass is

$$(\bar{x}, \bar{y}) = \left(0, \frac{3a}{2\pi}\right).$$

REMARK In mechanics courses, you will also see a "moment of inertia" which is a "second moment".

DEF (Moment of Inertia about the x-axis)

$$\boxed{I_x = \iint_D y^2 \rho(x,y)\, dA}$$

DEF Moment of Inertia about the y-axis:

$$\boxed{I_y = \iint_D x^2 \rho(x,y)\, dA}$$

And, we can define a moment of inertia about the origin:

DEF Moment of Inertia about the origin

$$\boxed{\begin{aligned} I_0 &= I_x + I_y \\ I_0 &= \iint_D (x^2+y^2)\rho(x,y)\, dA \end{aligned}}$$

EXERCISES

① Work example 4 in section 15.4 (8^{th} edition)

② Read about Radius of Gyration (15.4 8^{th} edition)