Lecture #3

15.5 (Webassign) / 15.4 (Book) Applications of double integrals

Say we have a region in a plane that is representative of a (flat) object:

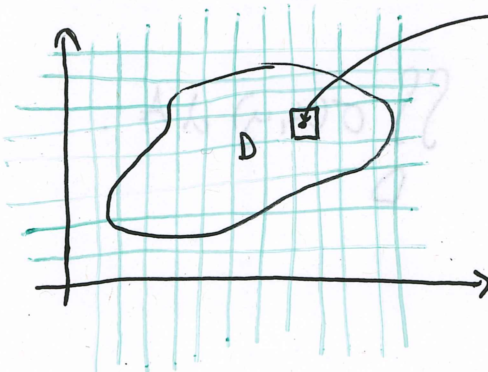
- We could compute the area of this region, $\iint_D dA$, giving us something in "units squared". Let's assume our units are meters, so m^2.
- If we had a "density" function, $\rho(x,y)$, which gives us the density at each point in D, we could compute the mass.

<u>Recall</u> (from physics?)

$$\text{Area} \cdot \overset{(\text{area})}{\text{density}} = \text{mass}$$

$$\left(m^2 \cdot kg/m^2 = kg \right)$$

- Think about Riemann sums:



$\rho(x_{ij}^*, y_{ij}^*)$

$$m = \lim_{k,\ell \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \rho(x,y) \cdot \underline{\Delta x \cdot \Delta y}$$

(area) density • area

Taking the limit, this converges to the integral:

$$\text{mass} = \boxed{m = \iint_D \rho(x,y)\, dA}.$$

<u>QUESTION</u> Did it really matter (in the construction of this integral) what the units were? Sure ... but not really!

Sure because we started with area, but no because we didn't end up with area ...

<u>What we did</u>: $\quad \text{area} \cdot \left(\frac{\text{mass}}{\text{area}}\right) = \underset{\text{mass}}{(\text{Total})}$

magic

<u>What we could do</u>: $\quad \text{area} \cdot \frac{(\text{ANY UNIT!})}{\text{area}} = \underset{(\text{ANY UNIT!})}{(\text{Total})}$

<u>EXAMPLE</u> Say we had the same region D as above, but instead of an area-density function $\rho(x,y)$, we were given a charge density (units of charge, Coulombs, per unit area), $\sigma(x,y)$.

charge density.

Then, if we wanted the <u>TOTAL CHARGE</u>, Q,

$$Q = \iint_D \sigma(x,y)\, dA.$$

QUESTION Given any region D with some area-density function $\rho(x,y)$, we can compute the total mass. [redacted] Can we find the center of mass? In other words, where we could put our finger to balance the "flat" plate D on our finger:

We can! This next part we will do "intuitively" ... we need two definitions.

DEF Moment about the x-axis

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^{*}\,\rho(x_{ij}^{*}, y_{ij}^{*})\,\Delta A$$

$$= \iint_D y\,\rho(x,y)\,dA \qquad (*)$$

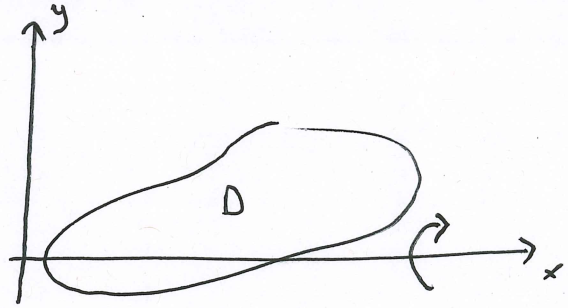
DEF Moment about the y-axis

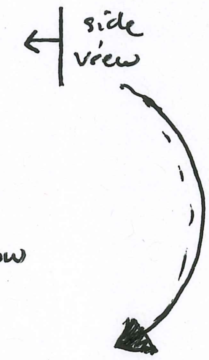
$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^{*}\,\rho(x_{ij}^{*}, y_{ij}^{*})\,\Delta A$$

$$= \iint_D x\cdot\rho(x,y)\,dA \qquad (*)$$

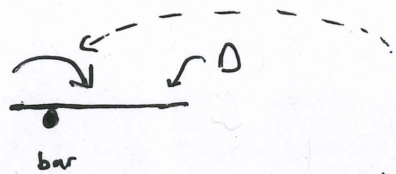
What are these "moments"?

(about the x-axis)





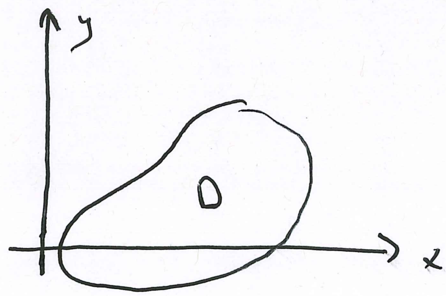
Place a "bar" beneath D along the x-axis. How unbalanced is it?



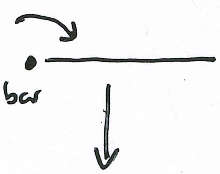
rotates this way!
unbalanced!

Notice how the distance from the bar (x-axis) is the y-coordinate!

(about the y-axis)



Place a "bar" beneath D along the y-axis.



(very unbalanced)

Notice how the distance from the bar (y-axis) to a point in D is just the x-coordinate!

Now we can define the center of mass:

<u>DEF</u> center of mass, $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x,y)\, dA$$

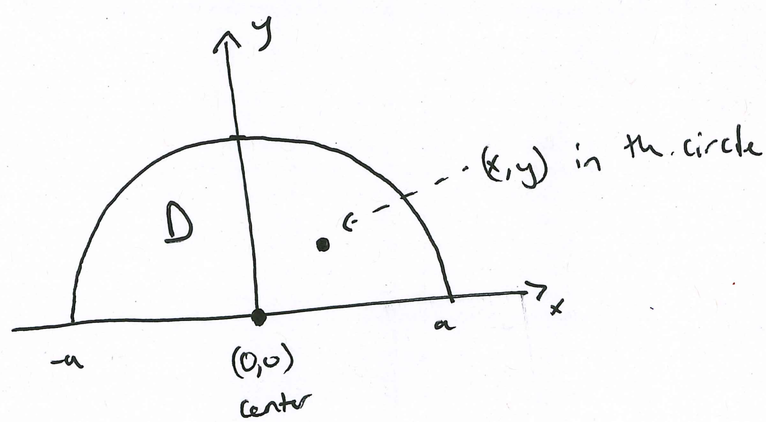
$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x,y)\, dA,$$

where m is the mass.

<u>REMARK</u> Notice that we are scaling the moment by mass!

<u>EXAMPLE</u> Consider the upper half of the circle $x^2+y^2=a^2$. Assume the density at any point in the half circle is proportional to the distance from the center of the circle. Find the center of mass.

Draw a picture!



<u>Density</u>: distance to center for any (x,y) in our region:

$$d = \sqrt{x^2+y^2}$$

~~distance~~ density is proportional to distance, so our density function is

$$\rho(x,y) = K\cdot \overset{\text{distance}}{d} = K\sqrt{x^2+y^2}$$

We need to compute the mass:

$$m = \iint_D \rho(x,y)\,dA = \iint_D K\sqrt{x^2+y^2}\,dA$$

use polar!

$r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2}$

$$= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a} (Kr)\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi} K\left[\frac{r^3}{3}\right]_{r=0}^{r=a} d\theta$$

$$= K\int_0^{\pi} \frac{a^3}{3}\,d\theta$$

$$= \frac{Ka^3}{3}\int_0^{\pi} d\theta = \frac{Ka^3}{3}\cdot\left[\theta\right]_0^{\pi}$$

$$= \boxed{\frac{K\pi a^3}{3}}$$

Now, we can compute $\bar{x}$.

<u>Exercise</u> Compute $\bar{x}$. Why is it 0? Did you need to compute $\bar{x}$ to know this?

And $\bar{y}$ ...

$$\bar{y} = \frac{1}{m}\iint_D y\cdot\rho(x,y)\,dA = \frac{3}{K\pi a^3}\int_0^{\pi}\int_0^{a}(r\sin\theta)\cdot(Kr)\cdot \overbrace{r\,dr\,d\theta}^{dA}$$

$$= \frac{3}{\pi a^3} \int_0^{\pi} \sin\theta \, d\theta \cdot \int_0^a r^3 \, dr$$

$$= \frac{3}{\pi a^3} \cdot \left[-\cos\theta \right]_0^{\pi} \cdot \left[\frac{r^4}{4} \right]_0^a$$

$$= \frac{3}{\pi a^3} \left[1 - -1 \right] \cdot \left(\frac{a^4}{4} \right)$$

$$= \frac{3}{\pi a^3} \cdot \frac{2a^4}{4}$$

$$= \frac{3a}{2\pi}.$$

Center of mass : $(\bar{x}, \bar{y}) = \left(0, \frac{3a}{2\pi}\right)$.

In mechanics courses, you will also see a "moment of inertia", which is a "second moment".

<u>DEF</u> (moment of Inertia about the x-axis)

$$\boxed{I_x = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \cdot \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D y^2 \rho(x,y) \, dA}$$

<u>DEF</u> (moment of inertia about the y-axis)

$$\boxed{I_y = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \cdot \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D x^2 \rho(x,y) \, dA}$$

We can also define the moment of inertia about the origin:

DEF Moment of inertia about the origin

$$I_0 = I_x + I_y$$

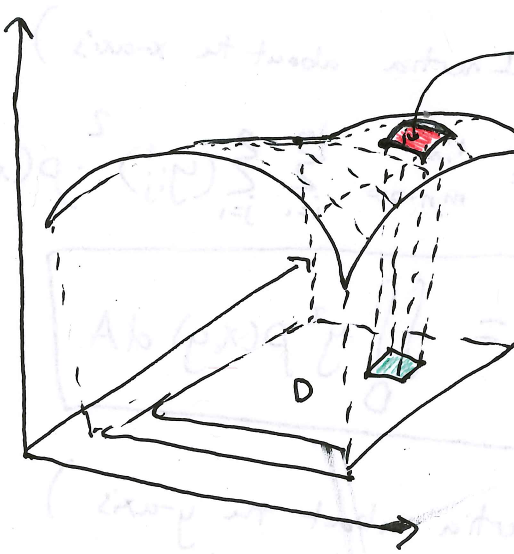
$$\boxed{I_0 = \iint_D (x^2+y^2)\rho(x,y)\, dA}$$

Exercise
- Work example 4 in section 15.4 (8th ed.)
- Read about Radius of Gyration

15.6 (Webassign) / 15.5 (Book - 8th Ed.) Surface Area!

REMARK We will revisit surface area in chapter 15, defining it with vector-valued functions.

Surface Area:

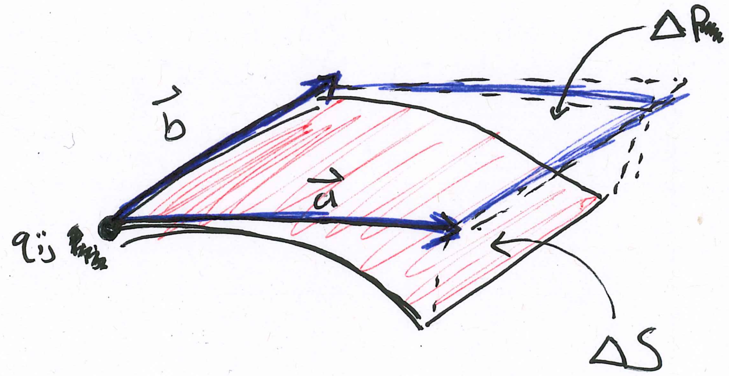


small piece of surface area...

Can we compute this?

Not really... but yes!

Look closely at our small piece of area:



rem $|a \times b| =$ ~~Area~~ Area of the parallelogram, ΔP !

We can approximate ΔS with ΔP. We need one thing though!

$\vec{a} = ?$

$\vec{b} = ?$

Exercise: What should $\vec{a}$ and $\vec{b}$ be? Notice, (it may not be clear in the drawing), we want $\vec{a}$ in the direction of the x-axis and $\vec{b}$ in the direction of the y-axis. So no change in the y-value for $\vec{a}$, no change in the x-value for $\vec{b}$.

Hint: What do you know about tangent planes to surfaces?

In class: $\vec{a} = \langle \Delta x, 0, f_x(q_{ij}) \cdot \Delta x \rangle$

$\vec{b} = \langle 0, \Delta y, f_y(q_{ij}) \cdot \Delta y \rangle$