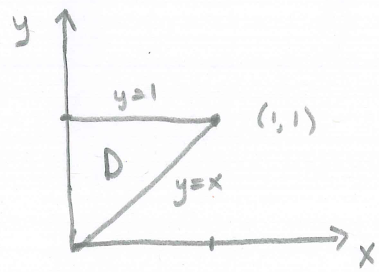
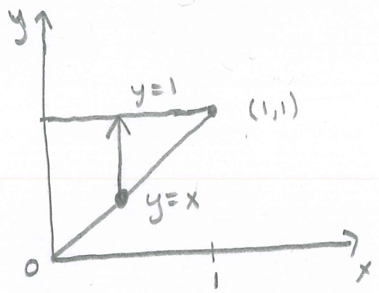
(15.2 continued)

EXAMPLES

① Write the integral $\iint_D f(x,y)\,dA$ in two ways, where D is as below:

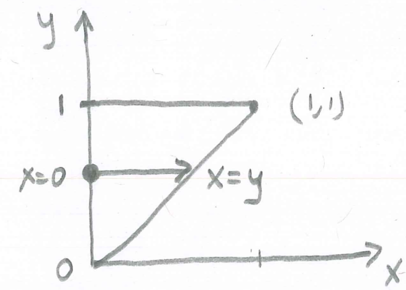


Method 1: integrating y first



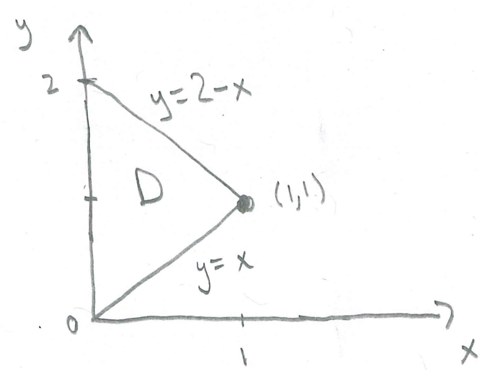
$$\int_{x=0}^{x=1}\int_{y=x}^{y=1} f(x,y)\,dy\,dx$$

Method 2: integrating x first

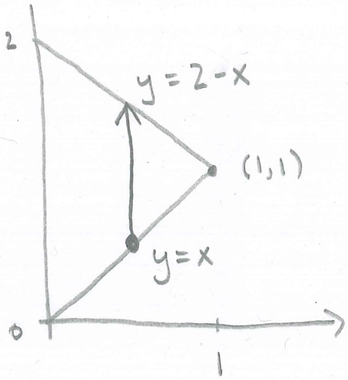


$$\int_{y=0}^{y=1}\int_{x=0}^{x=y} f(x,y)\,dx\,dy$$

② Same as above, but where D is

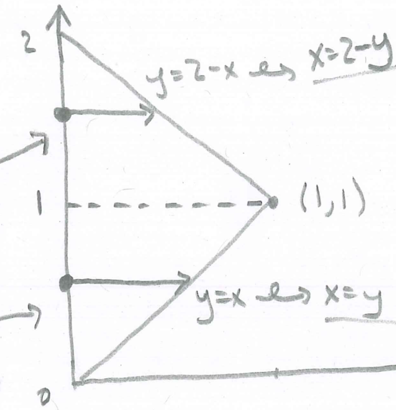


Method 1: Integrating y first



$$\int_{x=0}^{x=1}\int_{y=x}^{y=2-x} f(x,y)\,dy\,dx$$

Method 2: Integrating x first



careful! Breaks into two integrals!!

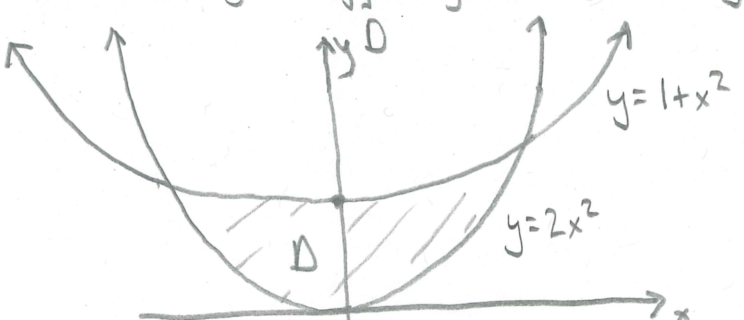
$$\int_{y=0}^{y=1}\int_{x=0}^{x=y} f(x,y)\,dx\,dy + \int_{y=1}^{y=2}\int_{x=0}^{x=2-y} f(x,y)\,dx\,dy$$

x goes from 0 to y only when y is between 0 and 1

similarly, x goes from 0 to $2-y$ only when y is between 1 and 2

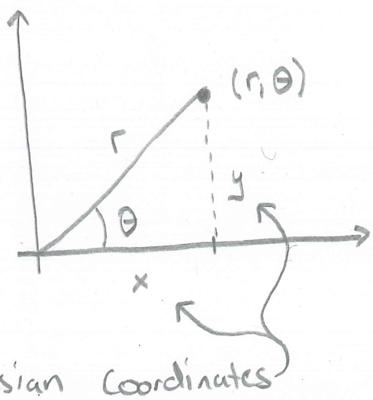
REMARK Notice that one of these methods is simpler. How can you make sure you are computing using the simplest method? Be careful about what variable you integrate first.

EXERCISE Write the integral $\iint f(x,y)\,dA$ in two ways, where D is:



15.3 Double Integrals in Polar Coordinates

RECALL: Polar Coordinates

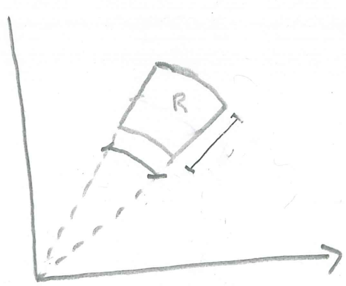


Notice: Cartesian Coordinates

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan(\theta) = \frac{y}{x} \end{cases} \quad \text{and} \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

QUESTION Why polar? Some regions are much easier to integrate over!

Ex



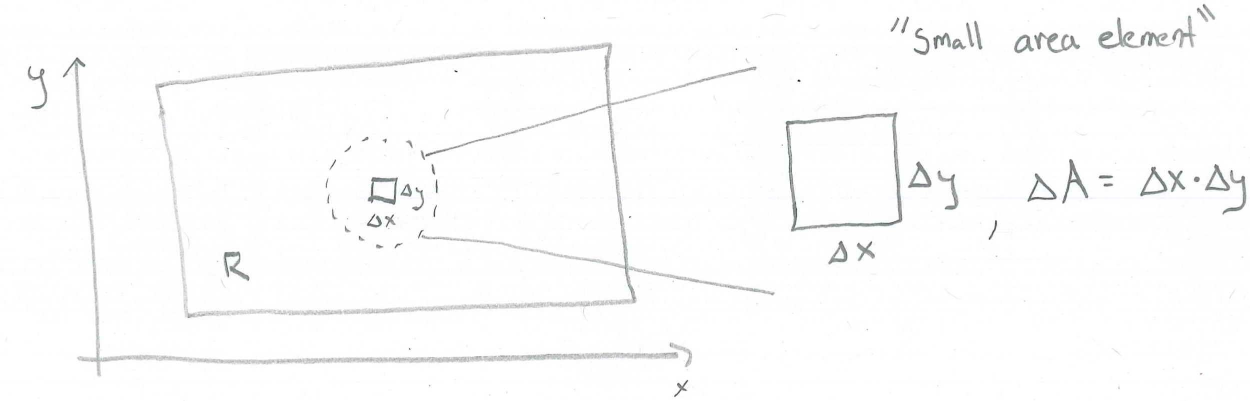
REMARK This figure is a "polar rectangle",

$$R = \{(r,\theta) \mid r_1 \le r \le r_2,\ \theta_1 \le \theta \le \theta_2\}.$$

Just as with Cartesian coordinates, we can integrate with bounds being functions.

QUESTION Is $\iint_R f(r,\theta)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$? NO!!!

REMARK Last time, we swept something under the rug. Let's go back to Cartesian coordinates for a moment.



$\Delta A = \Delta x \cdot \Delta y$

Riemann Sum : $\sum \underbrace{f(x^*, y^*)}_{\text{height}} \cdot \underbrace{\Delta A}_{\text{small area element}} = \sum f(x^*, y^*) \cdot \Delta x \cdot \Delta y$

If we let Δx and Δy get smaller and smaller (taking a limit)

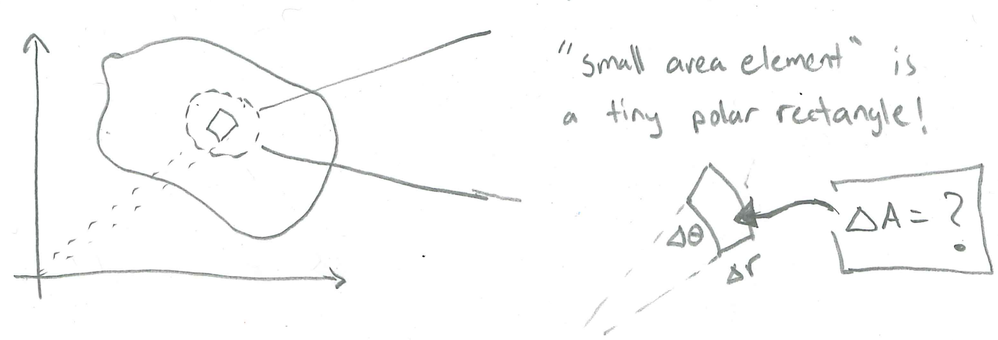
$$\begin{array}{ccc} \sum f(x^*, y^*) \, \Delta x \, \Delta y & \xrightarrow[\text{(limit)}]{} & \iint_D f(x,y) \, dx \, dy \\ \| & & \| \\ \sum f(x^*, y^*) \cdot \Delta A & \longrightarrow & \iint_D f(x,y) \, dA . \end{array}$$

So, we are saying that $\Delta A \rightarrow dA$, and $\Delta A = \Delta x \Delta y \rightarrow dx dy$,

i.e.

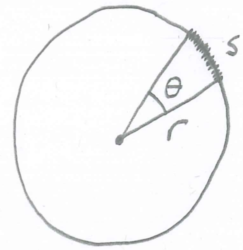
$$\boxed{dA = dx \, dy}$$

Now, we need to think about polar coordinates:



RECALL Arc length of a piece of a circle:

circumference :	$2\pi r$
arc length of s :	$\theta \cdot r$

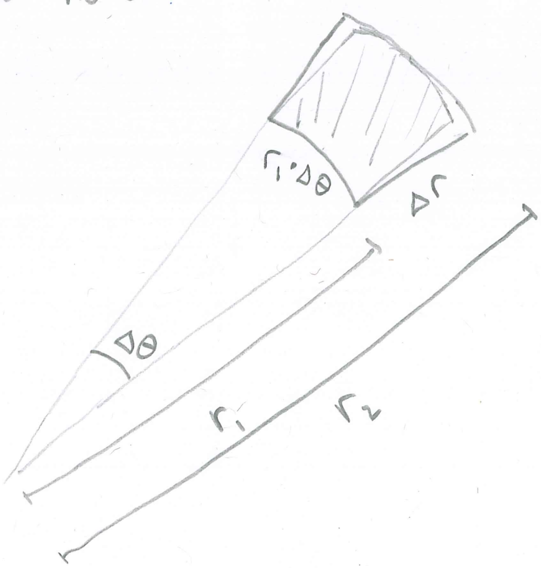


(notice, as θ goes to 2π, arc length of s approaches the circumference!)

So,

, and $\Delta A \approx r \Delta\theta \Delta r$

More precisely, let $\Delta r = r_2 - r_1$, i.e. r goes from r_1 to r_2, and notice



ΔA really close to

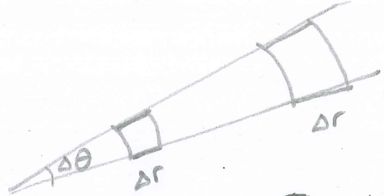
$r_1 \Delta\theta \cdot \Delta r$.

If r_1 and r_2 become closer and closer, $r_1 \to r_2$ and so we really only have one value of r in the limit...

So at each point (r, θ), when we take this limit letting the area element get smaller and smaller

$$\boxed{dA = r\,dr\,d\theta}$$

REMARK It makes sense that dA depends on r:



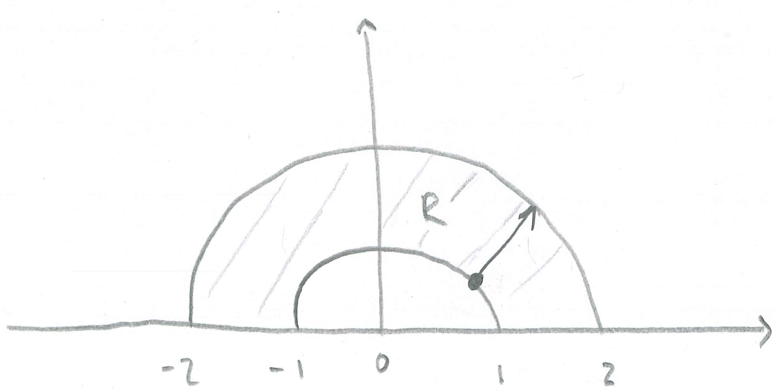
The bigger the r, the bigger the area element!

FORMULA

$$\boxed{\iint_D f(r,\theta)\,dA = \iint_D f(r,\theta)\, r\,dr\,d\theta}$$

EXAMPLE 1 Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Draw R!



Notice: this is a polar rectangle. r goes from 1 to 2 and θ goes from 0 to π.

Since this is a rectangular region, it doesn't matter which variable we integrate first!

$$\iint_R (3x+4y^2)\, dA = \int_{\theta=0}^{\theta=\pi} \int_{r=1}^{r=2} (3x+4y^2)\, r\,dr\,d\theta$$

$x = r\cos\theta$
$y = r\sin\theta$

$$= \int_{\theta=0}^{\theta=\pi} \int_{r=1}^{r=2} \left(3r\cos\theta + 4r^2\sin^2\theta\right) r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 3r^2\cos\theta\, dr\,d\theta + \int_0^{\pi}\int_1^2 4r^3\sin^2\theta\, d\theta$$

$$= \int_0^{\pi} \left[r^3\cos\theta\right]_{r=1}^{r=2} d\theta + \int_0^{\pi}\left[r^4\sin^2\theta\right]_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} 7\cos\theta\, d\theta + \int_0^{\pi} 15\sin^2\theta\, d\theta$$

trig identity!

$$= \left[7\sin\theta\right]_0^{\pi} + 15\cdot\frac{1}{2}\int_0^{\pi}(1-\cos 2\theta)\, d\theta$$

u-sub if you need

$$= 0 + \frac{15}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$= 0 + \frac{15}{2}\left((\pi - 0) - (0-0)\right)$$

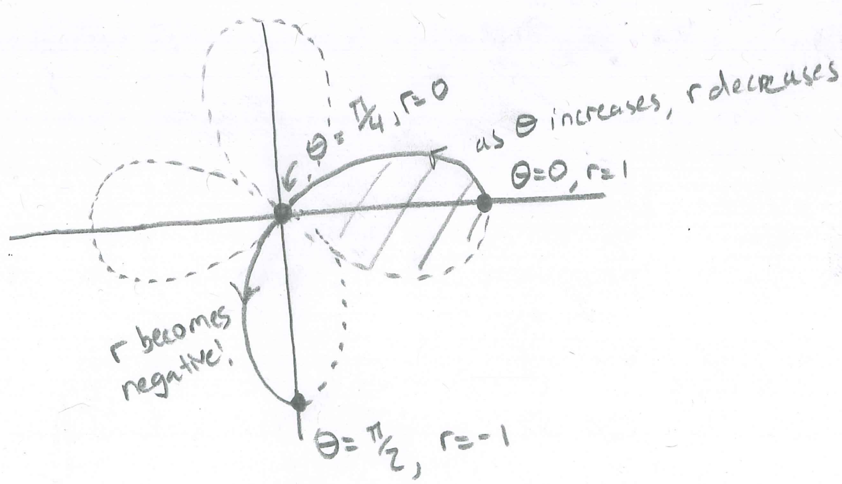
$$= \boxed{\frac{15\pi}{2}}$$

EXAMPLE 2 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

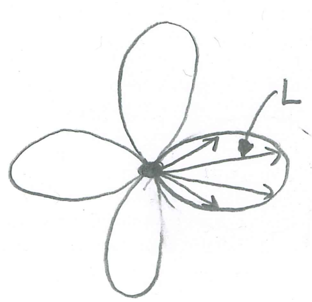
RECALL Property ⑤ of double integrals from Lecture #1 notes.

$$\boxed{\iint_R dA = \text{Area of } R} \quad !$$

STEP 1 Sketch out the curve $r = \cos 2\theta$. What are these 4 leaves? Notice that the curve repeats after π. Why?



STEP 2 To find the area of one leaf, write the domain of integration over a leaf:



$$A(L) = \iint_L dA,$$

and $L = \{(r,\theta) \mid -\pi/4 \le \theta \le \pi/4 ,\ 0 \le r \le \cos 2\theta\}$

STEP 3 Compute!

$$A(L) = \iint_L dA = \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \int_{r=0}^{r=\cos 2\theta} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \left[\frac{r^2}{2}\right]_{r=0}^{r=\cos 2\theta} d\theta$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta$$

(*) double angle identity

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \frac{1}{2}(1+\cos 4\theta)\, d\theta$$

u-sub if needed

$$= \frac{1}{4} \left[\theta + \frac{1}{4}\sin(4\theta)\right]_{\theta=-\pi/4}^{\theta=\pi/4}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4}+0\right) - \left(\frac{-\pi}{4}+0\right)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right)$$

$$= \boxed{\frac{\pi}{8}}$$

EXERCISES

1. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

2. Find the area of one loop of the rose $r = \cos 3\theta$.

3. Find the area of the region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$.

4. Review problem 40 in 15.3 in the 8th edition of the book.