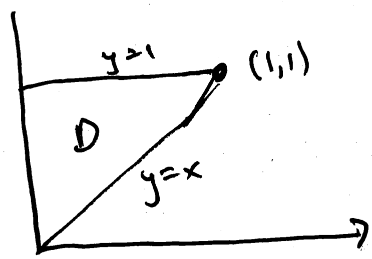
Lecture #2

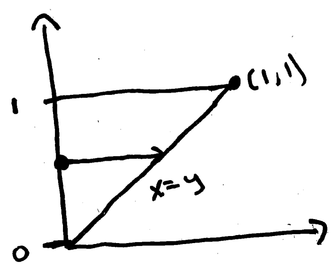
(15.3 cont.)

EXAMPLES

① Write the integral $\iint_D f(x,y)\,dA$ in two ways, where D is as below:

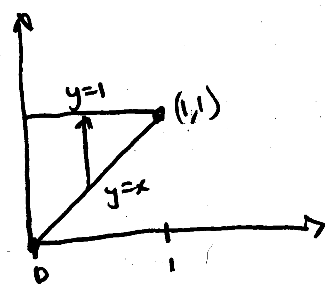


Integrating x first



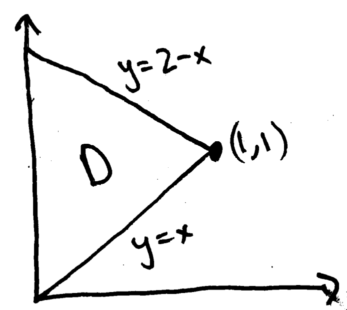
$$\int_{y=0}^{y=1}\int_{x=0}^{x=y} f(x,y)\,dx\,dy$$

Integrating y first

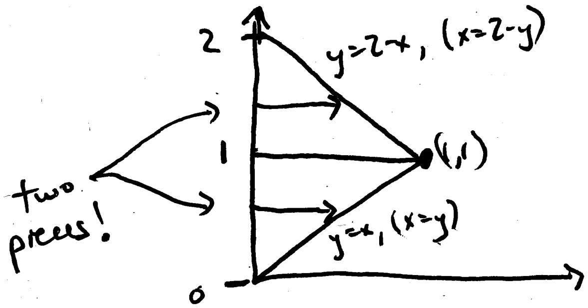


$$\int_{x=0}^{x=1}\int_{y=x}^{y=1} f(x,y)\,dy\,dx$$

② Same as above, but where D is

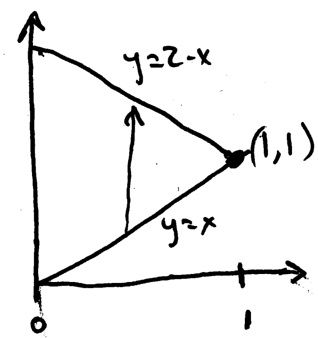


Integrating x first



$$\int_{y=0}^{y=1}\int_{x=0}^{x=y} f(x,y)\,dx\,dy + \int_{y=0}^{y=1}\int_{x=0}^{x=(2-y)} f(x,y)\,dx\,dy$$

Integrating y first

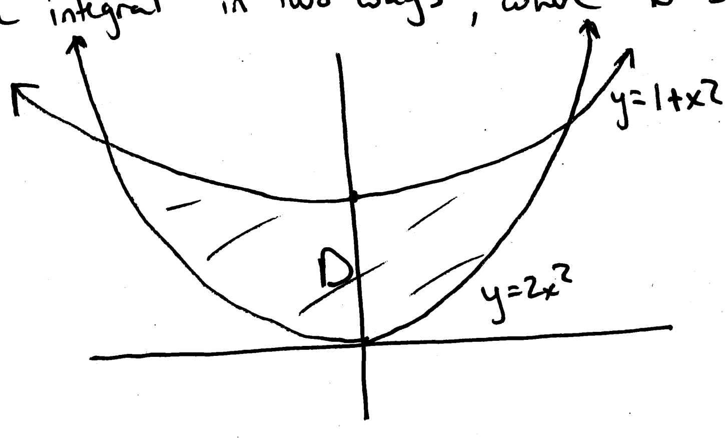


$$\int_{x=0}^{x=1}\int_{y=x}^{y=2-x} f(x,y)\,dy\,dx$$

* easier to compute! *

③ Exercise:

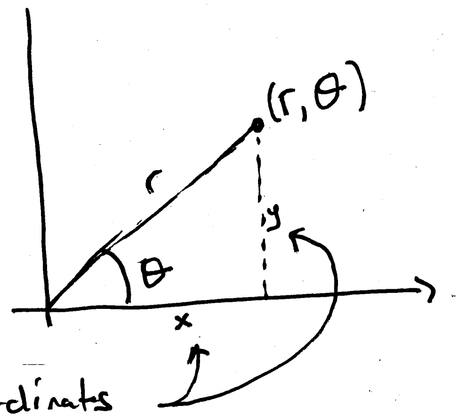
Write the integral $\iint_D f(x,y)\,dA$ in two ways, where D is:



REVIEW Properties of Double Integrals (from Lecture #1)

15.4 Double Integrals in Polar Coordinates

Rem Polar coordinates:

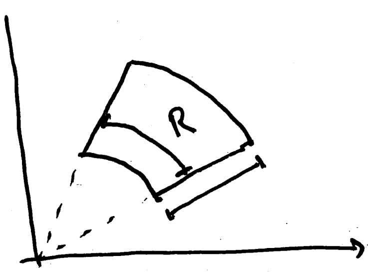


Notice: Cartesian coordinates

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan(\theta) = \frac{y}{x} \end{cases} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Why Polar? Some (regions) ~~domains~~ are much easier to integrate over!

Ex



In fact, this figure is a "polar rectangle",

$$R = \{ (r,\theta) \mid r_1 \le r \le r_2 \,,\ \theta_1 \le \theta \le \theta_2 \}.$$

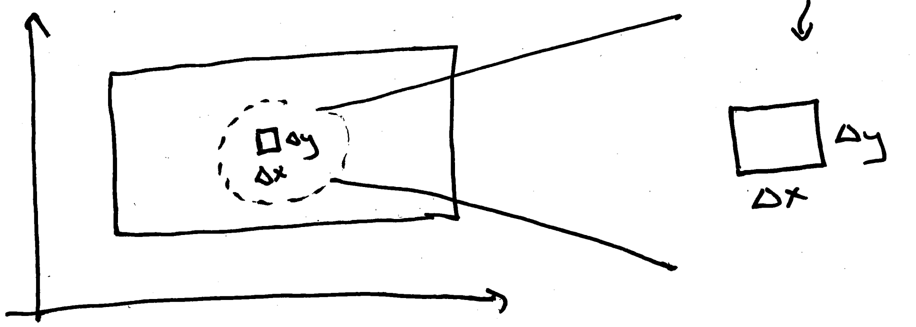
REMARK Just as with Cartesian coordinates, we can integrate with bounds being functions.

QUESTION Is $\displaystyle\iint_R f(r,\theta)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$?

NO!

REMARK Last time, we swept something under the rug. Returning to cartesian coordinates:

"small area element"



$\Delta A = \Delta x \cdot \Delta y$

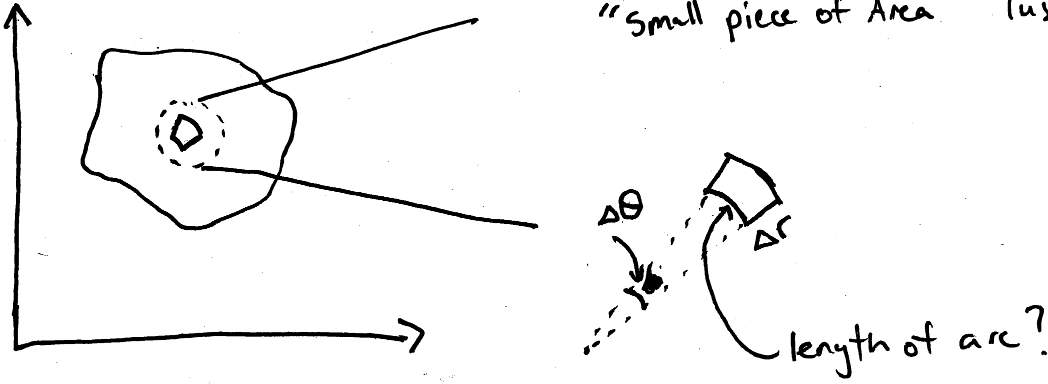
Recall "Riemann Sums": $\sum \underbrace{f(x^*, y^*)}_{\text{height}} \cdot \underbrace{\Delta x \, \Delta y}_{\Delta A}$

If we let Δx and Δy get smaller and smaller (taking a limit)

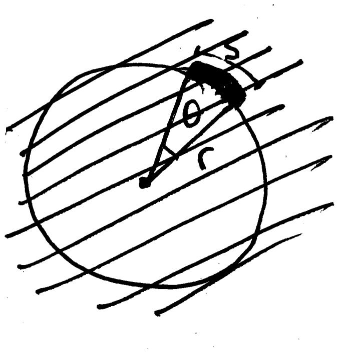
$$\iint_D f(x,y)\, dA = \iint_D f(x,y)\, dx\, dy.$$

Now, we need to think about polar coordinates!

"Small piece of Area" (use polar rectangles!)

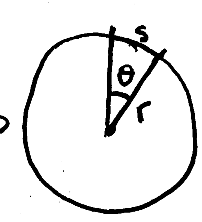


rem



circumference: $2\pi r$

arc length of s: θr, θ radians.



So:

, and $\Delta A \approx r\,\Delta\theta\,\Delta r$.

When we take a limit, we get exactly:

$$\boxed{dA = r\,dr\,d\theta}$$

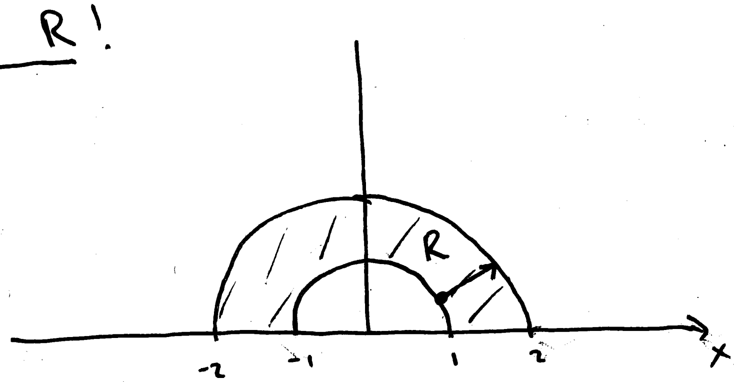
REMARK Later, we will make this rigorous!

Then,

$$\boxed{\iint_D f(r,\theta)\,dA = \iint_D f(r,\theta)\,r\,dr\,d\theta.}$$

EXAMPLE 1 Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Draw R!



Write integral: (rectangle, so we can integrate r first or θ first!)

$$\int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} (3x+4y^2)\, r\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} \left(3\cdot r\cos\theta + 4r^2\sin^2\theta\right) r\,dr\,d\theta$$

polar coordinates!

$$= \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} 3r^2\cos\theta\,dr\,d\theta + \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} 4r^3\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi} \left[r^3\cos\theta\right]_{r=1}^{r=2} d\theta + \int_0^{\pi} \left[r^4\sin^2\theta\right]_1^2 d\theta$$

$$= \int_0^{\pi} 7\cos\theta\,d\theta + \int_0^{\pi} 15\sin^2\theta\,d\theta$$

trig identity!

$$= \left[7\sin\theta\right]_0^{\pi} + 15\cdot\frac{1}{2}\int_0^{\pi}(1-\cos 2\theta)\,d\theta$$

u-sub

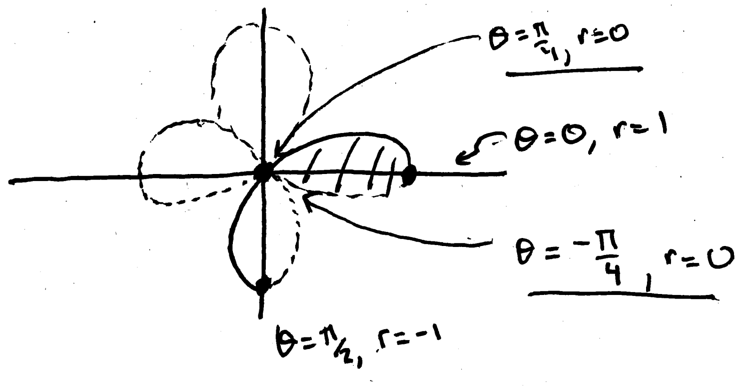
$$= 0 + \frac{15}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$= 0 + \frac{15}{2}\left((\pi-0)-(0-0)\right)$$

$$= \frac{15\pi}{2}.$$

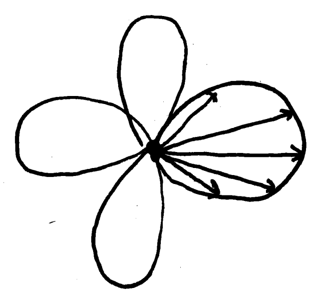
EXAMPLE 2 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Step 1 Sketch out the curve $r = \cos 2\theta$. What are these 4 leaves? Notice, repeats after π. Why?



· Leaf above :

$$L = \{ (r, \theta) \mid -\tfrac{\pi}{4} \le \theta \le \tfrac{\pi}{4} \,,\; 0 < r < \cos 2\theta \}$$



Integral :

$$A(L) = \iint_L dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{r^2}{2} \right]_0^{\cos 2\theta} d\theta$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \; d\theta$$

double angle!

$$= \frac{1}{2} \cdot \int_{-\pi/4}^{\pi/4} \frac{1}{2} (1 + \cos 4\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4}$$

u-sub

$$= \frac{1}{4}\left[\left(\frac{\pi}{4}+0\right) - \left(\frac{-\pi}{4}+0\right)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right)$$

$$= \boxed{\frac{\pi}{8}}$$

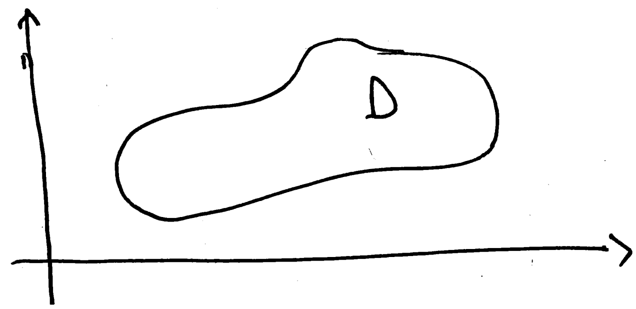
Exercises

- Find the volume of the solid that lies under the paraboloid $z = x^2+y^2$, above the xy-plane, and inside the cylinder $x^2+y^2=2x$
- Find the area of one loop of the rose $r = \cos 3\theta$.
- Find the area of the region inside cardioid $r = 1+\cos\theta$ and outside the circle $r = 3\cos\theta$.
- Review problem 40 in 15.3 in 8th edition of book.

15.5 Applications of Double integrals

Computing mass:

$$m = \iint_D \overbrace{\rho(x,y)}^{\text{density}}\, dA$$



Moment about x-axis:

$$M_x = \iint_D y\,\rho(x,y)\, dA$$

$$M_y = \iint_D x\,\rho(x,y)\, dA$$

Center of mass: $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\, dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\, dA$$