Lecture #1

15.1 Iterated Integrals (over rectangular domains)

REMARK If you would like a review of Riemann sums and estimation techniques for double integrals, read through pages 1028 - 1032 in the text. We will see Riemann sums a few times in the course.

EXAMPLE 1 Compute $\iint_R (x-3y^2)\,dA$, where $R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$.

Method 1: integrate with respect to (w.r.t) y first.

$$\int_{x=0}^{x=2}\int_{y=1}^{y=2} (x-3y^2)\,dy\,dx = \int_0^2 \left[\int_1^2 (x-3y^2)\,dy\right] dx$$

$$= \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx$$

$$= \int_0^2 \left[(2x-8)-(x-1)\right] dx$$

$$= \int_0^2 (x-7)\,dx$$

$$= \left[\frac{x^2}{2} - 7x\right]_{x=0}^{x=2}$$

$$= (2-14) - 0 = \boxed{-12}$$

Method 2: integrate with respect to (w.r.t.) x first.

$$\int_{y=1}^{y=2}\int_{x=0}^{x=2}(x-3y^2)\,dx\,dy = \int_1^2\left[\int_0^2 (x-3y^2)\,dx\right]dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3y^2x\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2-6y^2)\,dy$$

$$= \left[2y - 2y^3\right]_{y=1}^{y=2}$$

$$= (4-16) - (2-2)$$

$$= \boxed{-12}$$

<u>NOTE</u> Method 1 and Method 2 give the same answer! This suggests that there may be a theorem for rectangular domains along these lines... and there is!

<u>THM</u> (Fubini's Theorem)

If f is continuous on the rectangle $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$, then

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integral exists.

REMARK How do you use Fubini? Many ways! In this course, the most common application will be to change the order of integration. If you are having trouble integrating with respect to x first, try integrating with respect to y first!

EXAMPLE 2

$$\iint_R \sin(x)\cos(y)\,dA\,, \quad R = \underbrace{[0,\tfrac{\pi}{2}]}_{x} \times \underbrace{[0,\tfrac{\pi}{2}]}_{y}$$

$$= \int_0^{\pi/2} \left[\int_0^{\pi/2} \sin(x)\cos(y)\,dy \right] dx$$

Notice! $\sin(x)$ is a "constant" with respect to integration in the y-variable! ($\sin(x)$ does not depend on y.)

- So, we can pull the $\sin(x)$ out of the inner integral.

$$= \int_0^{\pi/2} \sin(x) \left[\int_0^{\pi/2} \cos(y)\,dy \right] dx$$

Now, notice that the inner integral is constant with respect to integration in x (it does not depend on x). So we can pull this entire integral out!

$$= \left[\int_0^{\pi/2} \cos(y)\,dy\right] \cdot \left[\int_0^{\pi/2} \sin(x)\,dx\right]$$

Now, we have taken a double (iterated) integral and turned it into two single-variable integrals multiplied!

EXERCISE Start with $\int_0^{\pi/2} \cos(y)\,dy \cdot \int_0^{\pi/2} \sin(x)\,dx$ and go backwards (make it a double integral).

QUESTION Under what circumstances can you take a double integral and turn it into two single integrals multiplied?

THM If $f(x,y) = g(x) \cdot h(y)$ is continuous on a rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \iint_R g(x) \cdot h(y)\,dA = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy.$$

QUESTION Is it okay if one of these integrals is improper?

EXAMPLE 3

$$\int_0^2 \int_0^{\pi} r^3 \cos^2(\theta)\,d\theta\,dr = \int_0^2 r^3\,dr \cdot \underbrace{\int_0^{\pi} \cos^2\theta\,d\theta}$$

(*) remember how to work this!

[Hint: trig identity ...]

EXERCISES

1. $\displaystyle\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\, dx$

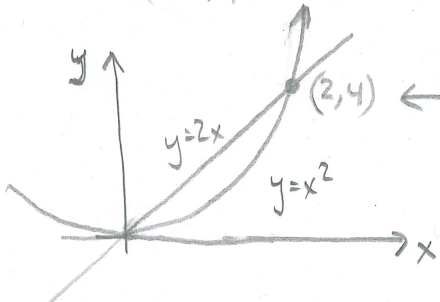
2. Find the volume of the solid lying under the elliptic paraboloid $\frac{x^2}{4} + \frac{y^2}{9} + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.

3. $\displaystyle\iint_R \frac{xy}{1+x^4}\, dA$, $R = \{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le 1\}$.

15.2 Double integrals over General Regions

EXAMPLE 1 Find the volume under $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

STEP 1 Draw D (!!!)

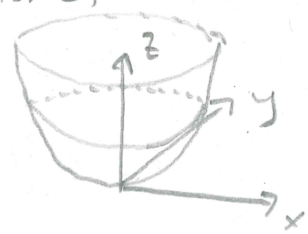


← compute point of intersection!

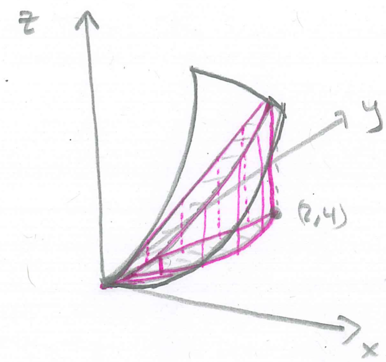
STEP 2 (optional... but good practice) Draw picture!

First, what is $z = x^2 + y^2$?

(paraboloid!)



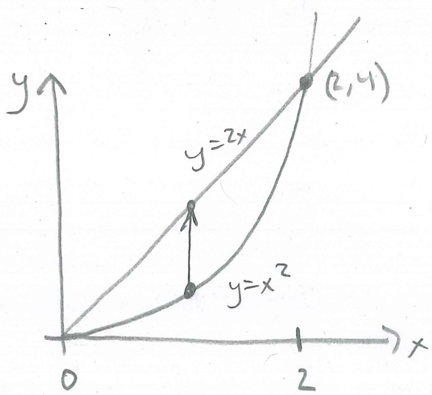
Now, draw the region above D:



try drawing this on your own

STEP 3 Choose whether to integrate w.r.t. x or y first.

(Method 1) Integrating with respect to y first:



Notice! Functions of x!

y goes from $\boxed{x^2 \text{ up to } 2x}$, while x goes from 0 to 2.

$$\int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} \underbrace{(x^2+y^2)}_{\boxed{z=f(x,y)}}\,dy\,dx = \int_0^2\left[\int_{x^2}^{2x}(x^2+y^2)\,dy\right]dx$$

$$= \int_0^2 \left[x^2y + \frac{y^3}{3}\right]_{x^2}^{2x} dx$$

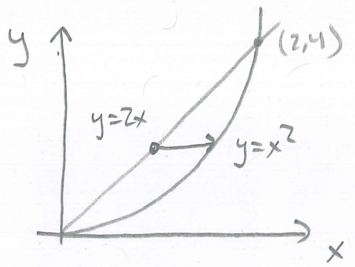
$$= \int_0^2 \left(2x^3 + \frac{8x^3}{3} - x^4 - \frac{x^6}{3}\right) dx$$

$$= \int_0^2 \left(\frac{14}{3}x^3 - x^4 - \frac{1}{3}x^6\right) dx$$

$$= \left[\frac{7x^4}{6} - \frac{x^5}{5} - \frac{x^7}{21} \right]_0^2$$

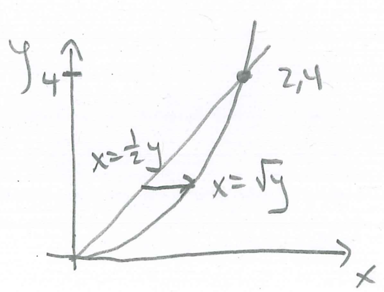
$$= \boxed{\frac{216}{35}}$$

(Method 2) Integrating with respect to x first:



So x goes from ? to ? ...

$\Rightarrow$ Need to solve $y=2x$ and $y=x^2$ for x!



$y=2x \rightarrow x=\frac{1}{2}y$

$y=x^2 \rightarrow x=\pm\sqrt{y}$

(*) choose $x=\sqrt{y}$

(why chose the positive root?)

So, we see:

x goes from $\frac{1}{2}y$ to $\sqrt{y}$,

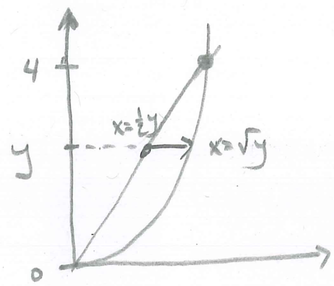
while y goes from 0 to 4.

Setting up the integral, we get

$$\int_{y=0}^{y=4} \left[\int_{x=\frac{1}{2}y}^{x=\sqrt{y}} (x^2+y^2)\, dx \right] dy.$$

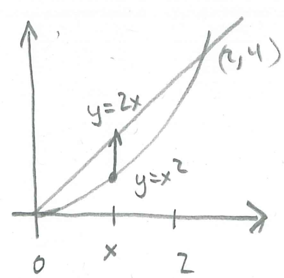
EXERCISE Work that last integral and confirm it gives you the same answer as Method 1.

REMARK It is sometimes easier to think "I want y to go from 0 to 4, so if I pick an arbitrary y between 0 and 4:



x goes from what to what?" Here, x should be a function of y.

Alternatively, if you want "x to go from 0 to 2"



then pick an arbitrary x between 0 and 2, and notice y goes from what to what?

REMARK This is giving us a way to extend Fubini to include general domains, not just rectangular domains! We still require the function to be continuous on the domain.

Properties of Double Integrals

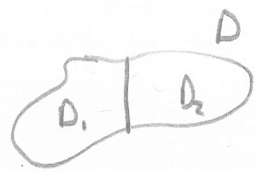
① $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$

② $\iint_D c \cdot f(x,y)\, dA = c \iint_D f(x,y)\, dA$, c constant

③ If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then

$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$$

④ If $D = D_1 \sqcup D_2$:



, then

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA.$$

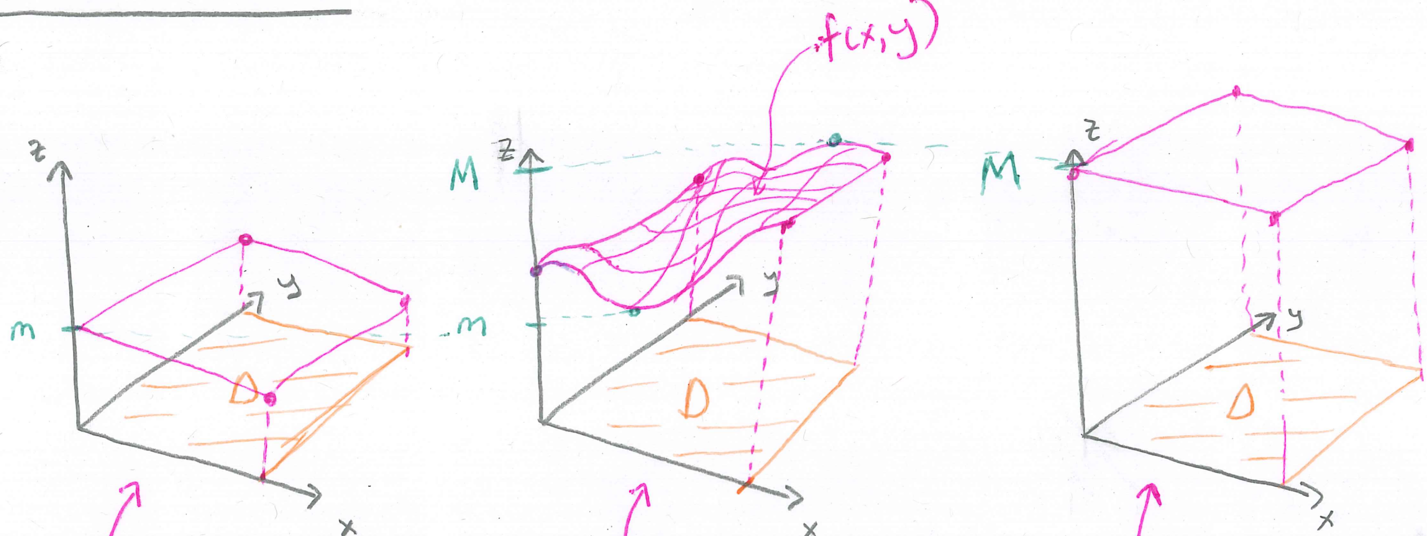
⑤ $$\iint_D dA = A(D)$$

<u>REMARK</u> Notice, $\iint_D dA = \iint_D 1 \cdot dA$, so we can think of $f(x,y) = 1$.

⑥ If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$m \cdot A(D) \le \iint_D f(x,y)\,dA \le M \cdot A(D).$$

INTERPRETATION



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$$\boxed{m \cdot A(D)} \quad \le \quad \boxed{\iint f(x,y)\,dA} \quad \le \quad \boxed{M \cdot A(D)}$$