Lecture #1 : 15.2, 15.3

15.2 Iterated Integrals (over rectangular domains)

EXAMPLE 1

$$\iint_R (x-3y^2)\,dA, \qquad R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$$

Method 1:

$$\underbrace{\int_{x=0}^{x=2}\int_{y=1}^{y=2} (x-3y^2)\,dy}\,dx = \int_0^2 \left[\int_1^2 (x-3y^2)\,dy\right] dx$$

$$= \int_0^2 \Big[xy - y^3\Big]_1^2\,dx$$

$$= \int_0^2 \big((2x-8)-(x-1)\big)\,dx$$

compare!

$$= \int_0^2 (x-7)\,dx$$

$$= \left[\frac{x^2}{2} - 7x\right]_0^2$$

$$= (2-14) - 0 = \boxed{-12}$$

Method 2:

$$\overbrace{\int_{y=1}^{y=2}\int_{x=0}^{x=2} (x-3y^2)\,dx}\,dy = \int_1^2 \left[\int_0^2 (x-3y^2)\,dx\right] dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3y^2x\right]_0^2 dy$$

$$= \int_1^2 (2-6y^2)\,dy$$

$$= \Big[2y - 2y^3\Big]_1^2 = (-12) - (2-2)$$

$$= \boxed{-12}$$

(Method 1 + Method 2 give the same answer!)

Fubini's Thm

If f is continuous on the rectangle $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$,

then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integral exists.

REMARK If you are having trouble integrating with respect to x first, try integrating with respect to y first!

~~THM~~

EXAMPLE 2

$$\iint_R \sin(x)\cdot\cos(y)\, dA, \qquad R = \overbrace{[0, \tfrac{\pi}{2}]}^{x} \times \overbrace{[0, \tfrac{\pi}{2}]}^{y}$$

$$= \int_0^{\pi/2} \left[\int_0^{\pi/2} \sin(x)\cdot\cos(y)\, dy \right] dx$$

$\sin(x)$ is a "constant" with respect to integration in the y-variable!

So, we can pull the $\sin(x)$ out of the inner integral.

$$= \int_0^{\pi/2} \sin(x) \cdot \left[\int_0^{\pi/2} \cos(y)\, dy \right] dx$$

But now, notice that the inner integral is constant with respect to integration in $\underline{\underline{x}}$! (There are $\underline{\underline{no}}$ x ~~vari~~ terms in the inner integral.) So, we can pull this $\underline{\underline{\text{whole}}}$ integral out!

$$= \left[\int_0^{\pi/2} \cos(y)\, dy \right] \cdot \left[\int_0^{\pi/2} \sin(x)\, dx \right].$$

Notice! ↗ this is now two single-variable integrals multiplied.

<u>REMARK</u> You could start with $\int_0^{\pi/2} \cos(y)\, dy \cdot \int_0^{\pi/2} \sin(x)\, dx$ and go backwards! (Make it a double integral :)

<u>THM</u> If $f(x,y) = g(x) \cdot h(y)$ is continuous on a rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\, dA = \iint_R g(x) \cdot h(y)\, dA = \int_a^b g(x)\, dx \cdot \int_c^d h(y)\, dy.$$

<u>Question worth pondering!</u>

Is it okay if these integrals are improper?

EXAMPLE 3

$$\int_0^2 \int_0^{\pi} r^3 \cos^2(\theta)\, d\theta\, dr = \underbrace{\int_0^2 r^3\, dr}_{} \cdot \underbrace{\int_0^{\pi} \cos^2(\theta)\, d\theta}_{\text{remember how to work this!}}$$

[Hint: Double angle formula ...]

~~EXAMPLE~~

EXERCISES

1. $$\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\, dx$$

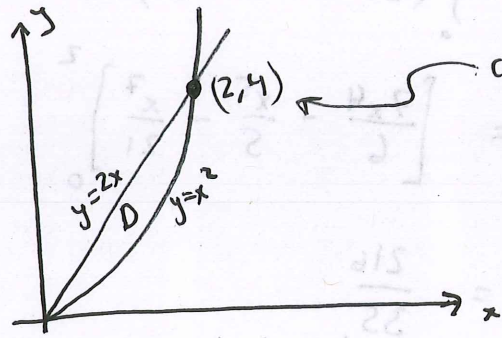
2. Find the volume of the solid lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the rectangle $R = [-1,1] \times [-2,2]$.

3. $$\iint_R \frac{xy}{1+x^4}\, dA, \quad R = \{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le 1\}.$$

15.3 Double integrals over General Regions

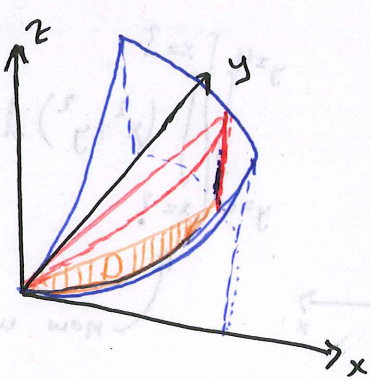
EXAMPLE 1 Find the volume under $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

STEP 1 Draw D!



compute point of intersection!

STEP 2 (optional) Draw picture!

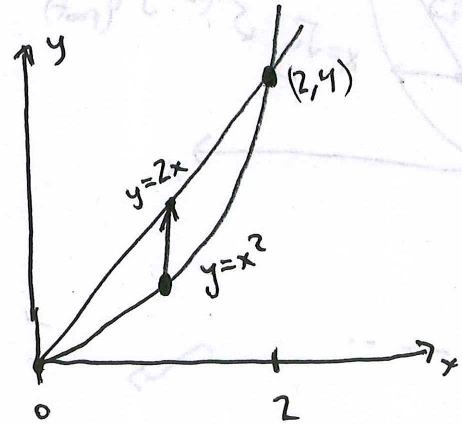


STEP 3 Choose whether to integrate w.r.t. x or y first.

Method 1

(Integrating w.r.t. y first)

with respect to



notice! functions!

$$\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} (x^2+y^2)\, dy\, dx$$

$$\int_0^2 \left[\int_{x^2}^{2x} (x^2+y^2)\, dy \right] dx = \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{x^2}^{2x} dx$$

$$= \int_0^2 \left[\left(2x^3 + \frac{8x^3}{3}\right) - \left(x^4 + \frac{x^6}{3}\right) \right] dx$$

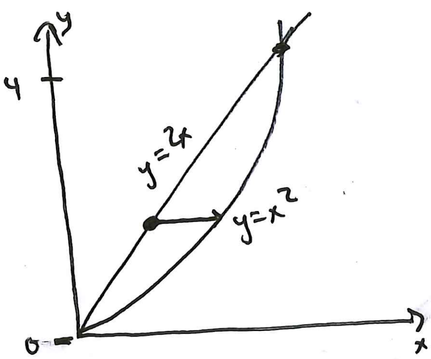
$$= \int_0^2 \left(\frac{14x^3}{3} - x^4 - \frac{x^6}{3} \right) dx$$

$$= \left[\frac{7x^4}{6} - \frac{x^5}{5} - \frac{x^7}{21} \right]_0^2$$

$$= \frac{216}{35}.$$

METHOD 2

(Integrating w.r.t. x first)

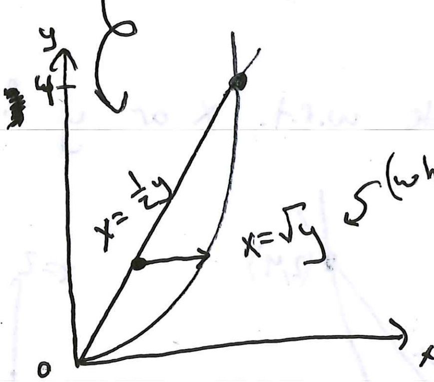


$$\int_{y=0}^{y=4} \left[\int_{x=?}^{x=?} (x^2+y^2)\, dx \right] dy$$

now we need $x(y)$ instead of $y(x)$!

$y = 2x$ solve for x $x = \frac{1}{2}y$

$y = x^2$ $x = \pm\sqrt{y}$ choose $x = \sqrt{y}$



(why positive? (root))

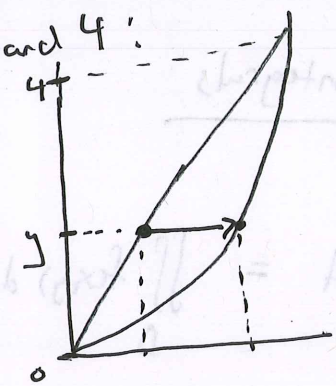
Then, - - - -

we get:

$$\int_{y=0}^{y=4}\left[\int_{x=\frac{1}{2}y}^{x=\sqrt{y}}(x^2+y^2)\,dx\right]dy.$$

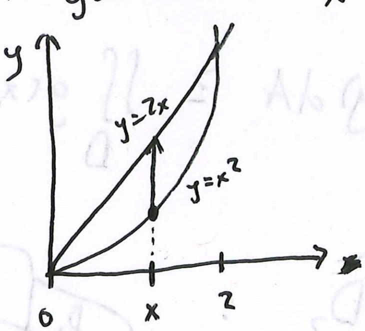
Work this to confirm you get the same answer!

REMARK It is sometimes easier to think "I want y to go from 0 to 4, so if I pick an arbitrary y between 0 and 4:



x goes from <u>what</u> to <u>what</u>? These should be functions of $\underline{\underline{x}}$.

Alternatively, if you want "x to go from 0 to 2"



then for an arbitrary x between 0 and 2, y goes from <u>what</u> to <u>what</u>? (Above, it is from $\underline{\underline{x^2}}$ to $\underline{\underline{2x}}$.)

REMARK This is giving us a way to extend Fubini to include general domains, not just rectangular domains! We still require the function to be continuous on the domain.

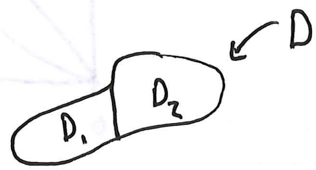
Exercise Do example 3 in section 15.3 in the book. Work the integral two ways. Is one way easier? Why?

Properties of Double Integrals

① $$\iint_D \left[f(x,y) + g(x,y)\right] dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA.$$

② $$\iint_D c \cdot f(x,y)\, dA = c \iint_D f(x,y)\, dA, \quad c \text{ constant}.$$

③ If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then
$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA.$$

④ If $D = D_1 \sqcup D_2$: 

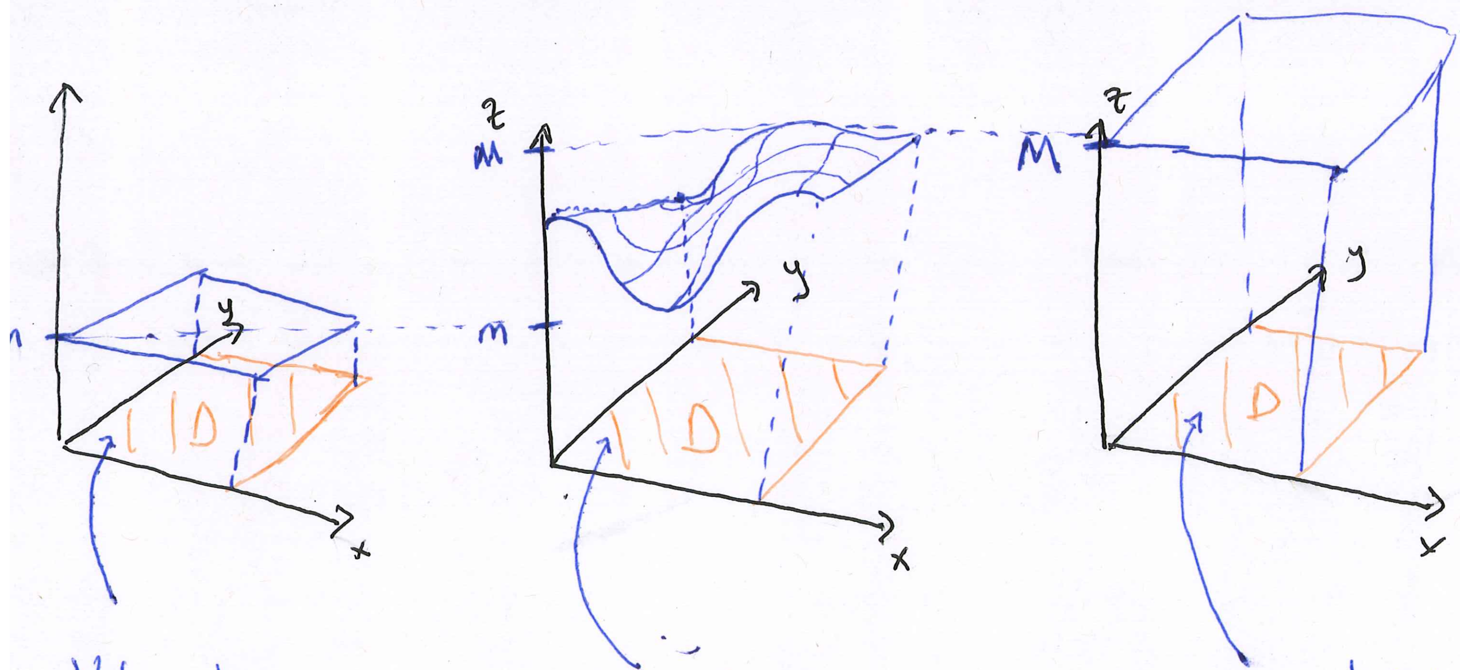
, then
$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

⑤ $$\iint_D dA = A(D)$$

(Notice, $\iint_D dA = \iint_D 1 \cdot dA$, so we can think of $f(x,y) = 1$.)

⑥ If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$m \cdot A(D) \le \iint_D f(x,y)\, dA \le M \cdot A(D).$$



Volume here: Volume here: Volume here:

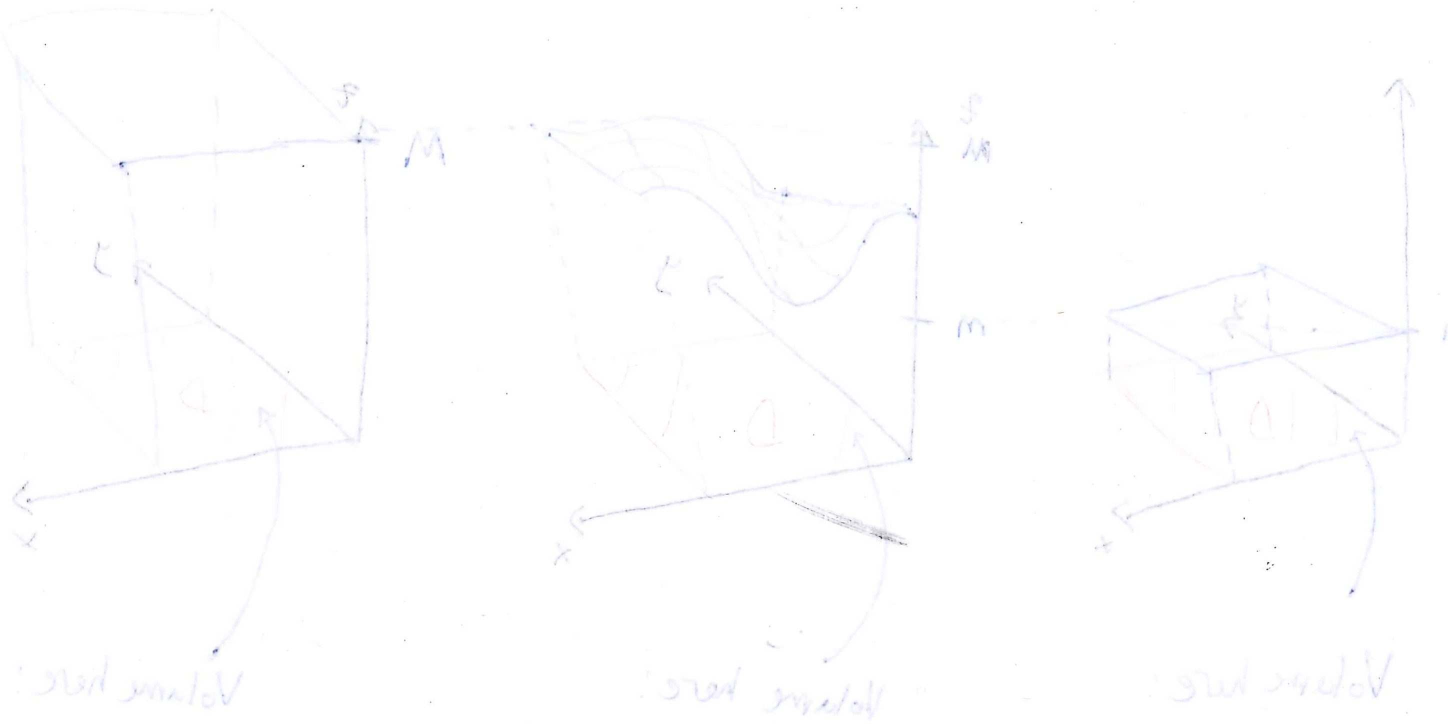
$$\boxed{m \cdot A(D)} \le \boxed{\iint_D f(x,y)\, dA} \le \boxed{M \cdot A(D)}$$

(5) $A(D) = \iint_D dA$

(Notice, $\iint_D dA = \iint_D 1 \cdot dA$, so we can think of $f(x,y) = 1$...)

(6) If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$m \cdot A(D) \le \iint_D f(x,y)\, dA \le M \cdot A(D)$$



$$\boxed{m \cdot A(D)} \le \boxed{\iint_D f(x,y)\, dA} \le \boxed{M \cdot A(D)}$$