16.6: Parameterized Surfaces

In 16.2, 16.3, and 16.4, we discuss integrals over curves which often involved parameterizing a curve. In 16.6, 16.7, 16.8, and 16.9, we will be discussing integrals over surfaces. But first we must discuss how to parameterize surfaces.

Recall $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ gives a parameterization for some curve C. As the one parameter t varies, (x, y, z) move along the curve. In a similar way, a surface S can be parameterized using two parameters in the form

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}.$$

Just as we did with one parameter problems, we can try to combine the equations $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$ in order to eliminate the parameters in hopes of getting an equation involving x, y, and z, so that we can identify the surface. We can also try to identify the surface, by looking at fixing u and considering the one parameter curve we get as v varies (and we can do the same by fixing v and varying u); we call these curves *grid curves*. A parameterized surface $\mathbf{r}(u, v)$ typically has a region D that describes all possible inputs (u, v), the output from D is some region on the surface.
In 16.7, 16.8 and 16.9, we often will need to find a parameterization and the corresponding domain for some surface. Here are some notes on how to find such a surface parameterization:

1. **Spheres**: We already know from spherical coordinates that $x^2+y^2+z^2 = a^2$ can be parameterized by

   $$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\mathbf{i} + a \sin\phi \sin\theta\mathbf{j} + a \cos\phi\mathbf{k}$$

   where ϕ and θ represent the same angles they did in spherical coordinates. So for the entire sphere the domain would be $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

2. **Cylinder**: By cylinder, I mean a situation where only two variables are present in the defining equation. Such as $y = f(x)$ (drawn in the xy-plane and extended out in the z-direction). We can parameterize $y = f(x)$ in $\mathbf{R}^3$ by

   $$\mathbf{r}(u, v) = u\mathbf{i} + f(u)\mathbf{j} + v\mathbf{k}$$

   The domain of u and v will be the domain restrictions given for x and z.

3. **Graphs**: By the *graph* of a two variable function, we mean a surface given by $z = f(x, y)$. We can parameterize $z = f(x, y)$ by

   $$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}$$

   The domain restriction for u and v will be the domain restrictions given for x and y.

4. **Surface of Revolution**: Consider the surface given by taking $y = f(x)$ on $a \le x \le b$ and rotating about the x-axis. If θ is the angle of rotation, we can parameterize using

   $$r(x, \theta) = x\mathbf{i} + f(x)\cos(\theta)\mathbf{j} + f(x)\sin(\theta)\mathbf{k}$$

   The domain restriction would be $a \le x \le b$ and $0 \le \theta \le 2\pi$ (for the full rotation).

5. **ADVICE**:

   (a) IMPORTANT: If you see $x^2 + y^2$ somewhere (unless, of course, a sphere), look to use the parameterization $x = u\cos(\theta)$, $y = u\sin(\theta)$, so that $x^2 + y^2$ would become u^2. In many situations it is smart to try this first. And you can do the same thing with $x^2 + z^2$ or $y^2 + z^2$ or even things like $4x^2 + y^2$ (on this last one use $x = \frac{1}{2}u\cos(\theta)$ and $y = u\sin(\theta)$). Typically, this simplifies things and the range of θ is usually 0 to 2π, but will depend on the problem. The range for u will need to be figured out from the rest of the problem.

   (b) A very special, but sometimes important, case is when you are given a point on a plane given by position vector $\mathbf{r}_0$, and two vectors parallel/along the plane $\mathbf{r}_1$ and $\mathbf{r}_2$ (not pointing in the same direction). This is situation we can get to any point on the plane by starting at $\mathbf{r}_0$ and walking u units along $\mathbf{r}_1$ and v units along $\mathbf{r}_2$. That is we can parameterize the plane via: $\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{r}_1 + v\mathbf{r}_2$.
   (But also note that if you get a plane in the standard form $ax + by + cz + d = 0$ and solve for z, then you can use the 'Graphs' method from above).

We then discussed surface area for the resulting surface, which will be important for all the remaining sections. And we found that if $\mathbf{r}(u, v)$ is a parameterized surface, S with the domain of D on u and v, then

$$A(S) = \text{ the surface area of } S \; = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$$

and we denote:

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

This is the comparable to the line integral differential notation $ds = |\mathbf{r}'(t)|dt$.

We computed $|\mathbf{r}_u \times \mathbf{r}_v|$ for some special cases and got:

1. **Spheres**: For $\mathbf{r}(\phi, \theta) = a\sin\phi\cos\theta\mathbf{i} + a\sin\phi\sin\theta\mathbf{j} + a\cos\phi\mathbf{k}$, we get

$$dS = a^2 \sin\phi \, dA.$$

2. **Graphs**: For $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$ we get

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dA.$$