## 16.1: Vector Fields

A **vector field** is a function that assigns a vector to each point in $\mathbb{R}^2$ or $\mathbb{R}^3$. Such a function is defined by **component functions** with the typical notation:

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j} \qquad \text{for } \mathbb{R}^2, \text{ and}$$

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k} \quad \text{for } \mathbb{R}^3.$$

1. **Notational Conventions:** When we are just discussing the function in general we may suppress the input variables and write $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$.

   We also sometimes prefer to identify a point $(x,y,z)$ with it's position vector $\mathbf{x} = \langle x,y,z\rangle$ and write the function in the form $\mathbf{F}(\mathbf{x}) = P(\mathbf{x})\,\mathbf{i} + Q(\mathbf{x})\,\mathbf{j} + R(\mathbf{x})\,\mathbf{k}$.

   We may also use the other forms for a vector interchangeably. That is, sometimes I will write, $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$. All of these are equivalent, just a matter of convenience.

2. **Visualizing a Vector Field**: We can 'plot' a vector field by choosing various sample points $(x,y)$, evaluation the vector function $\mathbf{F}(x,y)$ and drawing the vector from that point. With enough vectors plotted, we start to get a sense of the vector fields. If the functions are sufficiently complicated, we would need a computer to graph an accurate graph, but here are some ways we can get a sense of the vector field:

   Assume we are wish to visualize a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$

   (a) If it's easy, find when $P(x,y) = 0$. Wherever $P(x,y) = 0$, the vector is vertical. You might also find when $P(x,y) > 0$, the vectors will point toward the right, and when $P(x,y) < 0$, the vectors will point toward the left.

   (b) If it's easy, find when $Q(x,y) = 0$. Wherever $Q(x,y) = 0$, the vector is horizontal. You might also find when $Q(x,y) > 0$, the vectors will point upward, and when $Q(x,y) < 0$, the vectors will point downward.

   (c) Plot a few sample points. I'd say it's smart to plot at least 6-8 spread out points to get a good idea of what is going on.

   (d) Look at the formula for the length of the vectors which is $|\mathbf{F}| = \sqrt{P(x,y)^2 + Q(x,y)^2}$. Can you tell if the vectors are longer/shorter as you move in a particular direction.

3. **Particular Examples**

   (a) **Velocity Fields**: Consider moving through a fluid (or air), at each point we could measure the velocity of the fluid and plot this velocity as a vector. This would be a velocity field, $\mathbf{V}(x,y,z)$.

   (b) **Force Fields**: Consider the force vector acting on a particle at the point $(x,y,z)$. If we consider all these force vectors, this is a force field, $\mathbf{F}(x,y,z)$. Here are some notable force fields from physics:

   - *Gravitational Fields*: Newton's law of gravitation says that if $M$ is the mass of an object centered at the origin, $m$ is the mass of an object with location given by the position vector $\mathbf{x} = \langle x,y,z\rangle$, and $G$ is the gravitational constant, then the magnitude of the force between the objects is

$$|\mathbf{F}| = \frac{mMG}{|\mathbf{x}|^2}.$$

And the direction of this force on the object of mass $m$ is toward the origin, which would be the direction $-\mathbf{x}$ (as a unit vector it would be $-\frac{1}{|\mathbf{x}|}\mathbf{x}$). Multiplying this direction by the length from Newton's law gives

$$\mathbf{F} = \frac{mMG}{|\mathbf{x}|^3}\mathbf{x} = \left\langle \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

That is the gravitational field.

- *Electrical Fields*: Similarly for electrical charges, if $Q$ is a charge at the origin, $q$ is another charge with location given by $\mathbf{x}$, $\epsilon$ is a particular constant, then by Coulomb's Law the force on the the charge $q$ is given by

$$\mathbf{F} = \frac{\epsilon q Q}{|\mathbf{x}|^3}\mathbf{x}$$

For like charges we have repulsion with $qQ > 0$, and for opposite charges we have attraction with $qQ < 0$. Instead of considering the electrical force, physicists often consider the force per unit charge $\mathbf{E} = \frac{1}{q}\mathbf{F} = \frac{\epsilon Q}{|\mathbf{x}|^3}\mathbf{x}$ which is sometimes called the *electric field* of Q.

- *Gradient Fields*: If $z = f(x,y)$ or $w = f(x,y,z)$ are scalar functions, then their gradients define vectors at each point. So the gradient of a scalar function is a vector field. We say such a field is a *gradient field*. Any field that is a gradient field of some scalar function, that is $\mathbf{F} = \nabla f$, is said to be a **conservative vector field** and we call $f$ a *potential function* for $\mathbf{F}$).

So in order for a vector field to be a gradient field, it must be the gradient of some scalar function. **No all vector fields are gradient fields!** In fact if you randomly selected functions $P(x,y)$ and $Q(x,y)$, the chances of it being a scalar field are very small. However, many important vectors fields are gradient fields. And if we do have a gradient field, then there are many nice properties and simplifications we can use as will be discussed in 6.3 and 6.4 (and we will explain where the words *conservative* and *potential* are coming from).

Note: The gravitational field is a gradient field. One potential function is
$f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$, because if you compute the gradient of this function you get

$$\nabla f = \left\langle \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}} \right\rangle = \text{ gravitational force field}$$

Similarly the electrical field is a gradient field. So we say these fields are conservative.

4. **Bonus Discussion about Flow Lines**: I think a natural question to ask is the following: If a particle is placed in a velocity/force field at a particular point, then what path will it follow. Let's explore what we can say here (I'll focus on $\mathbb{R}^2$, but a similar discussion would follow in $\mathbb{R}^3$):

   (a) *Flow Lines for Velocity Fields*: Assume a velocity field is given $\mathbf{V}(x, y) = \langle P(x, y), Q(x, y)\rangle$. And we want to find the parametric equations for the path of a particle $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Then we need the velocity vectors for the particle to be equal to the velocity vectors of the field, so we need:

$$\frac{dx}{dt} = P(x, y) \text{ and } \frac{dy}{dt} = Q(x, y)$$

   which means we have a system of differential equations to solve.
   If we just want the equation for the path, then we can try to solve the differential equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q(x, y)}{P(x, y)}$$

   Here are some examples (I encourage you to look at the vector fields and convince yourself that these paths are correct).

   - If the velocity field is $\mathbf{V}(x, y) = \langle -y, x\rangle$, then the path of a particle would satisfy $\frac{dy}{dx} = \frac{x}{-y}$. This is separable, so we can solve to get $-\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$, which can be simplified to $x^2 + y^2 = K$ for some constant $K$. So in this force field particle follow the paths of circles.
   - If the velocity field is $\mathbf{V}(x, y) = \langle x, -y\rangle$, then the path of a particle would satisfy $\frac{dy}{dx} = \frac{-y}{x}$. This is separable, so we can solve to get $y = \frac{K}{x}$ for some constant $K$. So in this force field particle follow the paths that look like $y = \frac{1}{x}$.

   (b) *Flow Lines for Force Fields*: Assume a force field is given $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$. And we want to find the parametric equations for the path of a particle $\mathbf{r}(t) = \langle x(t), y(t)\rangle$.
   By Newton's second law, $\mathbf{F}(x, y) = m\mathbf{a}(x, y)$. So the acceleration of the particle at a given point is $\mathbf{a}(x, y) = \frac{1}{m}\mathbf{F}(x, y)$.
   We need the acceleration vectors for the particle to be equal to the acceleration vectors from the field, so we need:

$$\frac{d^2x}{dt^2} = \frac{1}{m}P(x, y) \text{ and } \frac{d^2y}{dt^2} = \frac{1}{m}Q(x, y)$$

   which means we have a system of second order differential equations to solve. This can get messy quick and even with simple examples we may want a computer to give us approximations for the path.
   Here is one very simple example:

   - Assume a particle of mass 2 kg is only acted on by the force of gravity so the force field (a 2D projection) is $\mathbf{F}(x, y) = \langle 0, -2(9.8)\rangle$ and acceleration is $\mathbf{a}(x, y) = \langle 0, -9.8\rangle$
     We need to solve the second order differential equation

$$\frac{d^2x}{dt^2} = 0 \text{ and } \frac{d^2y}{dt^2} = -9.8$$

     And you get $x(t) = at + b$ and $y = -4.9t^2 + ct + d$, for some constants $a$, $b$, $c$, and $d$ (you should have seen these equations before).

In any case, you can find the flow lines in either situation by solving differential equations.