15.8, 15.9 Review

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems. Hopefully this review sheet will remind you of some of the key ideas of these sections.

15.8: Cylindrical Coordinates

1. This is useful when working on a problem that is symmetric about an axis (Cylinders, Cones, Paraboloids, Spheres). In other words, when polar coordinates will work nice in two of the variables.

   In such a situation, if we have a choice, we typically pick z as the axis of symmetry, then we use polar coordinates in the typical way in x and y. We call this cylindrical coordinates.

2. If P is a point in 3-dimensions, let O be the origin, and Q be the point you get when you projection P on the xy-plane (the point directly 'below' P). Then the cylindrical coordinate representation is (r, θ, z), where

   $r = |\overline{OQ}| =$ distance from the origin to Q.
   $\theta =$ angle counterclockwise that $\overline{OQ}$ makes with the positive x-axis.
   $z =$ height above the xy-plane.

   We can convert from cartesian to cylindrical coordinates using the relationships:

   $$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = z, \qquad \text{and} \qquad x^2 + y^2 = r^2.$$

3. Here are some common curves given in cylindrical coordinates (here c is a constant):

   $r = c \Leftrightarrow x^2 + y^2 = c^2 \Leftrightarrow$ 3D circular cylinder of radius c symmetric about z-axis.
   $\theta = c \Leftrightarrow y = \tan(c)x \Leftrightarrow$ vertical plane making angle c with xz-plane.
   $z = r^2 \Leftrightarrow z = x^2 + y^2 \Leftrightarrow$ circular paraboloid symmetric about z-axis.
   $z = r \Leftrightarrow z^2 = x^2 + y^2 \Leftrightarrow$ circular cone symmetric about z-axis.

4. To evaluate a triple integral in cylindrical coordinates:

   (a) Find the upper and lower bounds for z. ($h_1(x, y) \leq z \leq h_2(x, y)$)

   (b) Replace $x = r\cos(\theta)$ and $y = r\sin(\theta)$ everywhere.

   (c) Describe the projection of the solid onto the xy-plane in polar coordinates. ($\alpha \leq \theta \leq \beta$ and $g_1(\theta) \leq r \leq g_2(\theta)$)

   (d) Replace $dV = r dz dr d\theta$, and integrate!

   $$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r\cos(\theta),\ r\sin(\theta))}^{h_2(r\cos(\theta),\ r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)\ r dz dr d\theta.$$

15.9: Spherical Coordinates

1. This is useful when working on a problem that is symmetric about a point (Cones and Spheres). We often find ourselves integrating over a sphere, in which case this method is very helpful. And that is why we devote a whole section to it!

2. If P is a point in 3-dimensions, again let O be the origin, and Q be the point you get when you projection P on the xy-plane. Then the spherical coordinate representation is (ρ, θ, ϕ), where

   $\rho = |\overline{OP}| =$ distance from the origin to P.
   $\theta =$ angle counterclockwise that $\overline{OQ}$ makes with the positive x-axis.
   $\phi =$ angle between 0 and *pi* that $\overline{OP}$ makes with the positive z-axis.

   For simplicity and clarity, we restrict $\rho \geq 0$ and $0 \leq \phi \leq \pi$.

   If we look at the right triangles these angles create, we can quickly write down some relationships and derive the equations relating (x, y, z) to (ρ, θ, ϕ). Let me do that quickly just to remind you of how that goes (I mention this here, because this is how I remember it).

   We already have $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The angle ϕ makes a right triangle with hypotenuse ρ, adjacent side z and opposite side r. Thus, we have $r = \rho\sin(\phi)$ and $z = \rho\cos(\phi)$. And all together we find:

   $$x = \rho\sin(\phi)\cos(\theta), \qquad y = \rho\sin(\phi)\sin(\theta), \qquad z = \rho\cos(\phi), \qquad \text{and} \qquad x^2 + y^2 + z^2 = \rho^2.$$

3. Here are some common curves given in spherical coordinates (here c is a constant):

   $\rho = c \Leftrightarrow x^2 + y^2 + z^2 = c^2 \Leftrightarrow$ sphere of radius c.
   $\theta = c \Leftrightarrow y = \tan(c)x \Leftrightarrow$ positive vertical plane making angle c with xz-plane.
   $\phi = c \Leftrightarrow z = \tan(c)(x^2 + y^2) \Leftrightarrow$ half circular cone (depending on $c < \pi/2$ or $c > \pi/2$).

4. To evaluate a triple integral in spherical coordinates:

   (a) Find the upper and lower bounds for ϕ, θ, and ρ. (Remember what they represent)

   (b) Replace $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, and $z = \rho\cos(\phi)$.

   (c) Replace $dV = \rho^2\sin(\phi)d\rho d\theta d\phi$, and integrate!

   $$\iiint\limits_E f(x, y, z)\ dV = \int_\gamma^\delta \int_\alpha^\beta \int_a^b f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\ \rho^2\sin(\phi)d\rho d\theta d\phi.$$