MATH 308 N

Final Exam

December 12, 2019

Name ____

Student ID #____________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________________

SOLUTIONS !

1	9	
2	14	
3	10	
4	10	
5	8	
6	18	
7	11	
Bonus	5	
Total	80	

- Your exam should consist of this cover sheet, followed by 7 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 110 minutes to complete the exam and there are 7 problems. Try not to spend more than 15 minutes on each problem.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

EXTRA FORMULAS:

Area of a Square: Area = base · height

Area of a Parallelogram: Area = base · height (or) $|u \times v|$

Length (Magnitude) of a vector in $\mathbb{R}^2$: Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2}.$$

Dot product in $\mathbb{R}^2$: Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$. Then

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = v_1 w_1 + v_2 w_2.$$

Angle between two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$:

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|\,|\vec{w}|}$$

1. (9 Points) True / False and Short Answer.

   Clearly indicate whether the statement is true or false. **Justify your answer.** You may cite properties from theorems.

   (a) **TRUE** / **FALSE** A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ can be one-to-one.

   - Since $4 > 3$, we have a theorem that says T cannot be one-to-one.
   - Alternatively, $\underbrace{\dim(\text{col}(A))}_{\text{rank}} \leq 3$, for A s.t. $T(\vec{x}) = A\vec{x}$. That means, by rank-nullity, that $\text{nullity}(A) \geq 1$. Since $\text{nullity}(A) \neq 0$, the associated linear transformation cannot be one-to-one.

   (b) **TRUE** / **FALSE** Let A be a 3×3 matrix whose eigenvalues are 3 of multiplicity 1 and 7 of multiplicity 2. Then A is invertible.

   - $\lambda = 0$ is not an eigenvalue, hence A is invertible by the unifying theorem.

   (c) **TRUE** / **FALSE** If an $n \times n$ matrix A has an eigenvalue $\lambda = 0$, then the row space of A is equal to $\mathbb{R}^n$.

   - Since $\lambda = 0$ is an eigenvalue, $\text{row}(A) \neq \mathbb{R}^n$ by the unifying theorem.

2. (14 Points) Construct an example and short answer.

Give an example of each of the following. If there is no such example, write NOT POSSIBLE. You do not need to justify solutions to parts (a) and (b), but you do need to justify work where specified in part (c).

(a) **Give an example** of a 3×3 matrix A such that $\det(A) = 0$ and $\operatorname{rank}(A) = 2$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) **Give an example** of a 2×4 matrix A with $\operatorname{nullity}(A) = 2$ and

$\operatorname{col}(A) = \operatorname{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{lin. ind.}} \right\}.$

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

or

2 pivot columns.

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \text{ b/c } \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

(c) **Fill in the blank** Assume $p(\lambda) = (\lambda+2)(\lambda-1)^2(\lambda+1)^2$ is the characteristic polynomial of a matrix B. Then

i. B is a __5__ $\times$ __5__ matrix.

ii. The eigenvalues of B are __$-2, 1, -1$__

iii. Is B invertible? **Justify your answer.**

yes, $\lambda = 0$ is not an eigenvalue.

iv. Is B guaranteed to be diagonalizable? If so, justify your answer. If not, explain what you would need to know to guarantee B is diagonalizable.

No, it is not. To be diagonalizable, we would need to know two things:

① $\dim(E_1) = 2$

and

② $\dim(E_{-1}) = 2$

← we need the dimension of the eigenspaces to match the multiplicity of the corresponding eigervalues.

3. (10 Points)

(a) Clearly circle the sets of vectors below that permit a linear combination equal to $\begin{bmatrix} a \\ b \end{bmatrix}$ for any a, b real numbers. You are not required to show work on this part. You may use your geometric intuition.

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \end{bmatrix} \right\},$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ -10 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\},$$

(b) Assume you circled the correct sets in part (a). What is the span of each of the circled sets?

$\mathbb{R}^2$

(c) Which of the circled sets are linearly independent sets? **Justify your answer.**

$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \end{bmatrix} \right\}$ are linearly independent by the Unifying Theorem

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$ is <u>not</u> because there are <u>three</u> vectors in $\mathbb{R}^2$.

(d) Which of the non-circled sets are linearly independent sets? **Justify your answer.**

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is lin. ind. by definition: $\left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \end{array} \right]$ has <u>one</u> solution.

$\underbrace{\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\}}$ and $\underbrace{\left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ -10 \end{bmatrix} \right\}}$.

$\hookrightarrow 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ $\qquad \hookrightarrow -2\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ -10 \end{bmatrix}$.

$\Rightarrow$ we see both sets are lin. dep. because one of the vectors is in the span of the remaining vectors

4. (10 points) Let $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ be a set of vectors in $\mathbb{R}^4$ where

$$\vec{u}_1 = \begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix}2\\1\\1\\1\end{bmatrix}, \vec{u}_3 = \begin{bmatrix}4\\1\\3\\5\end{bmatrix}$$

(a) It turns out S is a linearly dependent set. Show this by writing $\vec{u}_3$ as a linear combination of $\vec{u}_1$ and $\vec{u}_2$, then explain why this shows S is linearly dependent.

$$\left[\begin{array}{cc|c}1&2&4\\0&1&1\\1&1&3\\2&1&5\end{array}\right] \sim \left[\begin{array}{cc|c}1&2&4\\0&1&1\\0&-1&-1\\0&-3&-3\end{array}\right] \sim \left[\begin{array}{cc|c}1&2&4\\0&1&1\\0&0&0\\0&0&0\end{array}\right] \Rightarrow \begin{array}{l} x_2 = 1 \\ x_1 + 2x_2 = 4 \\ \Leftrightarrow x_1 = 2 \end{array}, \text{ so } \vec{x} = \begin{bmatrix}2\\1\end{bmatrix},$$

in other words: $\boxed{2\vec{u}_1 + \vec{u}_2 = \vec{u}_3}$

(b) The set S does not span $\mathbb{R}^4$. You may take $\{\vec{u}_1, \vec{u}_2\}$ as a basis for the span of the set S. Expand this basis to a basis for $\mathbb{R}^4$.

$$\text{Let } S' = \left\{ \begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}. \quad (\text{notice: } \mathrm{Span}(S') = \mathbb{R}^4.)$$

We use method 2 to generate a basis for $\mathbb{R}^4$:

$$\begin{bmatrix}1&2&1&0&0&0\\0&1&0&1&0&0\\1&1&0&0&1&0\\2&1&0&0&0&1\end{bmatrix} \sim \begin{bmatrix}1&2&1&0&0&0\\0&1&0&1&0&0\\0&-1&-1&0&1&0\\0&-3&-2&0&0&1\end{bmatrix} \sim \begin{bmatrix}1&2&1&0&0&0\\0&1&0&1&0&0\\0&0&-1&1&1&0\\0&0&-2&3&0&1\end{bmatrix}$$

$$\sim \begin{bmatrix}1&2&1&0&0&0\\0&1&0&1&0&0\\0&0&-1&1&1&0\\0&0&0&1&-2&1\end{bmatrix} \Rightarrow \mathcal{B}_{\mathbb{R}^4} = \left\{ \begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \right\}.$$

$\uparrow \ \uparrow \ \uparrow \ \uparrow$

pivot columns

5. (8 Points) You saw a similar problem on the first midterm. With our new machinery from Chapter 5, try to find an easier way to solve it.

   (a) Find all values of a such that the set of vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is **linearly dependent**.

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -10 \\ a \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 2 \\ a \\ -5 \end{bmatrix}.$$

   HINT: Notice that there are **three** vectors in the set, and the vectors are in $\mathbb{R}^3$.

3 vectors in $\mathbb{R}^3$

Let $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$. By the unifying theorem, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent if and only if $\det(A) = 0$ (A is not invertible).

$$\det\left(\begin{bmatrix} 1 & 1 & 2 \\ 2 & -10 & a \\ -3 & a & -5 \end{bmatrix}\right) = 50 - 3a + 4a - 60 + 10 - a^2 = a - a^2$$

Solve: $a - a^2 = 0$

$a(a-1) = 0$

$\boxed{a = 0,\ a = 1}$

   (b) Can you use part (a) to give the values of a such that the set spans $\mathbb{R}^3$? If so, what are they? Explain your thinking.

For the set to span $\mathbb{R}^3$, by the unifying theorem it must also be a linearly independent set. A set is either linearly independent or linearly dependent, so since for $a=0$ or $a=1$ we have a linearly dependent set, we can conclude that for any $a \neq 0$ or 1, we have a linearly independent set. Hence, for $a \neq 0$ or 1, the set will span $\mathbb{R}^3$.

6. (18 points)

(a) Find the all eigenvalues of the matrix $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$.

$$0 = \det(A - \lambda I_2) = \det\left(\begin{bmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{bmatrix}\right) = (9-\lambda)(6-\lambda) - 4 = 54 - 6\lambda - 9\lambda + \lambda^2 - 4$$
$$= \lambda^2 - 15\lambda + 50$$
$$= (\lambda - 10)(\lambda - 5).$$

$\boxed{\text{The eigenvalues are } 5, 10.}$

(b) Find an eigenvector corresponding to each eigenvalue of the matrix.

$\lambda = 5$: $(A - 5I_2)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = s$
$x_1 = \frac{1}{2}s$ $\Rightarrow$ $\vec{v} = s\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$\boxed{\text{eigenvector: } \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$ let $s=2$ (no fractions is nice)

$\lambda = 10$: $(A - 10I_2)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = s$
$x_1 = -2s$ $\Rightarrow$ $\vec{v} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\boxed{\text{eigenvector: } \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$

(c) Express each eigenspace as of the span of a set of vectors.

$\lambda = 5$: $E_5 = \text{span}\left\{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right\}$

$\lambda = 10$: $E_{10} = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

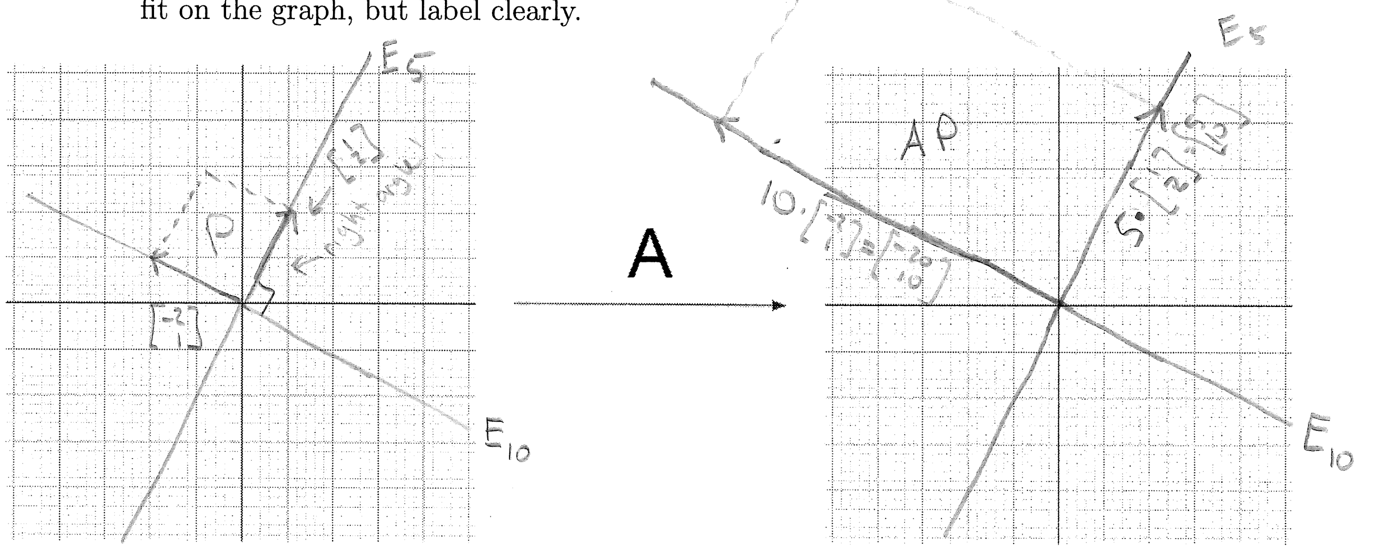
(d) Is the matrix diagonalizable? If so, express the matrix in the form UDU^{-1} for some matrices U and D where D is a diagonal matrix. If not, justify.

Let $U = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, then $U^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}$

and $D = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$, so

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}}$$

(e) This part has many steps. On the graph on the left: 1) draw and label the eigenspaces, 2) add the eigenvectors you chose and label them, and 3) draw a parallelogram with two sides being the eigenvectors. On the graph on the right: 1) draw the eigenspaces, and 2) draw the image of the parallelogram. You may scale vectors as needed so they fit on the graph, but label clearly.



(f) Compute the area of the parallelogram (on the left graph).

Method 1: $\left\|\begin{bmatrix}-2\\1\end{bmatrix}\right\| = \sqrt{4+1} = \sqrt{5}$, $\left\|\begin{bmatrix}1\\2\end{bmatrix}\right\| = \sqrt{5}$, so $\text{Area}(P) = 5$ (right angle between vectors!)

Method 2: $\left\|\begin{bmatrix}-2\\1\end{bmatrix} \times \begin{bmatrix}1\\2\end{bmatrix}\right\| = \left\|\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 1 & 0\\ 1 & 2 & 0\end{vmatrix}\right\| = |-5\hat{k}| = 5.$

(g) Compute the area of the image of the parallelogram (on the right graph).

Method 1: $\left\|\begin{bmatrix}5\\10\end{bmatrix}\right\| = \sqrt{125}$, $\left\|\begin{bmatrix}-20\\10\end{bmatrix}\right\| = \sqrt{500}$, so $\text{Area}(AP) = (\sqrt{125})(\sqrt{500})$
$= \sqrt{(125)(500)}$
$= \sqrt{62500}$
$= \underline{250}$

Method 2: $\left\|\begin{bmatrix}5\\10\end{bmatrix} \times \begin{bmatrix}-20\\10\end{bmatrix}\right\| = \left\|\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & 10 & 0\\ -20 & 10 & 0\end{vmatrix}\right\| = |-250\hat{k}| = 250$

(h) How is the change in area related to the eigenvalues?

Area scales = product of the eigenvalues.
$\quad\parallel \qquad\qquad\qquad \parallel$
50. ← $5 \cdot \underline{50} = 250$ $\qquad 5 \cdot 10$

(i) Compute the determinant of D (the diagonal matrix from part d) and show it is the same as the determinant of A, and describe how the determinant is related to change in area.

$\det(D) = 50$
$\det(A) = 54 - 4 = 50$

This is also the factor by which the area scales.

7. (11 points) Let $\vec{v} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$. Let $\mathscr{B} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ be a non-standard basis for $\mathbb{R}^2$.

(a) Compute $[\vec{v}]_{\mathscr{B}}$. Note that $\vec{v}$ is a vector in the standard basis.

$[\vec{v}]_{\mathscr{B}} = U^{-1}\vec{v}$, where $U = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, so $U^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$$[\vec{v}]_{\mathscr{B}} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -5 \end{bmatrix}}.$$

(b) Let $\mathscr{C}$ be another basis such that the change of basis matrix **from $\mathscr{C}$ to $\mathscr{B}$** is $\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}$. In other words, one would apply this matrix to a vector written in the basis $\mathscr{C}$ to get the same vector written in the basis $\mathscr{B}$. Find $[\vec{v}]_{\mathscr{C}}$.

$\mathscr{C}$ to $\mathscr{B}$: $\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}$. Need to go <u>from $\mathscr{B}$ to $\mathscr{C}$</u>: $\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}^{-1} = 4\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}$

so: $$[\vec{v}]_{\mathscr{C}} = \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}[\vec{v}]_{\mathscr{B}} = \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ -5 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

(c) What is the basis $\mathscr{C}$? **Justify your answer.**

We pass through the standard basis and recognize $\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}$ as a product of two matrices, one of which contains information about the basis for $\mathscr{C}$:

V ← columns of V give the basis $\mathscr{C}$!

$$\begin{tikzcd} \mathscr{C} \arrow[r, "V"] \arrow[rr, bend right, "\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}"'] & \text{Std.} \arrow[r, "U^{-1}"] & \mathscr{B} \end{tikzcd}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{2} & 1 \end{bmatrix} = VU^{-1}$$

$\Leftrightarrow$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}U = V, \text{ so } \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \rightsquigarrow \boxed{\mathscr{C} = \left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \right\}}$$

BONUS: (5 points) Assume A and B are diagonalizable, square matrices and that A and B have the same eigenspaces. Show that $AB = BA$. Be sure to **justify your work.**

A and B are diagonalizable: $A = U D_A U^{-1}$

$B = V D_B V^{-1}$.

A and B have the same eigenspaces, so we can pick a basis of eigenvectors for A that is also a basis of eigenvectors for B. Since U and V consists of columns that are exactly this basis of eigenvectors, we can pick U to be the same as V! Hence:

$$A = U D_A U^{-1}$$

$$B = U D_B U^{-1}$$

Then, computing:

$$\begin{aligned} AB &= U D_A U^{-1} U D_B U^{-1} \\ &= U D_A D_B U^{-1} \\ &= U D_B D_A U^{-1} \\ &= U D_B U^{-1} U D_A U^{-1} \\ &= BA. \end{aligned}$$

$\star$ because D_A and D_B are diagonal, and diagonal matrices commute!

Thus, $AB = BA$. ☒