Math 308G - Winter 2018
Final
2018-03-13

Name: ____
Student ID Number: ________________________________

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
Total:	72	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 2 is the same for each part.
- I treat row and column vectors as the same.
- For any linear transformation T, there exists a matrix A such that $T(x) = Ax$. I defined the determinant, rank, and nullity of T using A. This means,

$$\det(T) = \det(A), \quad \text{rank}(T) = \text{rank}(A), \quad \text{nullity}(T) = \text{nullity}(A).$$

1. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE". You do not need to justify your answers.

   (a) (2 points) If possible, give an example of a 2×2 matrix A that is not diagonalizable but A^2 is diagonalizable.

   (b) (2 points) If possible, give an example of a 2×2 matrix A such that $A^2 = I_2$ and $\text{nullity}(A) = 1$.

   (c) (2 points) If possible, give an example of a 2×2 matrix with distinct eigenvalues that is not invertible.

   (d) (2 points) If possible, give an example of linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\text{rank}(T) = \text{rank}(S) = \text{rank}(T \circ S) = 1$.

   (e) (2 points) If possible, give an example a 2×4 matrix A such that $\text{rank}(A) = \text{nullity}(A)$.

   (f) (2 points) If possible, give an example of a 2×2 matrix A such that 1 is not an eigenvalue of A^2 but 1 is an eigenvalue of A^4. (Think geometrically).

2. Perform the following computations.

(a) (6 points) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}.$$

- What is the reduced echelon form of A?

- Does $Ax = (1, 2, 1)$ have a solution? If so, give the general solution.

- Does $Ax = (1, 1, 1)$ have a solution? If so, give the general solution.

- What is a basis for row(A)?

(b) (6 points) Let

$$B = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- What is B^{-1}?

- What is the charateristic polynomial of B^2? (Hint: Be careful about the sign.)

3. Let A and B be equivalent matrices given by

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 1 & 1 & 3 \\ 2 & 5 & 6 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

Denote the columns of A by a_1, a_2, a_3, a_4.

(a) (3 points) Give a basis for null(A).

(b) (3 points) Give the general solution to $Ax = a_2$.

(c) (3 points) Give the general solution to $2Ax - a_3 = a_1 + a_2$.

$$(1/2, 1/2, 1/2, 0) + s_1(-3, -8, 5, 1)$$

(d) (3 points) It turns out $e_1 = (1, 0, 0, 0) \notin \text{col}(A)$. Give a vector v such that $v \neq e_1$ and $Ax = e_1 - v$ has a solution.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformations given by

$$T(x, y, z) = (x + 2z, y + z, x + y + z, z)$$

and

$$S(x, y, z) = (2x + y + z, y + z, y + z)$$

(a) (2 points) There exists a matrix A such that $T(v) = Av$. What is A?

(b) (2 points) There exists a matrix B such that $S(v) = Bv$. What is B?

(c) (4 points) These should be quick questions.

- What is $\text{rank}(S)$?
- What is $\det(S)$?
- What is $\text{nullity}(2S)$?
- What are 2 different eigenvalues of S?

(d) (4 points) Recall that $(T \circ S)(v) = T(S(v))$.

- What is the rank of $T \circ S$?
- What is a basis for $\text{range}(T \circ S)$? (Hint: Look at the 2nd two columns of B).

5. Let

$$A = \begin{bmatrix} 0 & -1 & 3 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

be a matrix which decomposes as $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Let u_1, u_2, u_3, u_4 be the columns of U and $\mathcal{B} = \{u_1, u_2, u_3, u_4\}$.

(a) (6 points) Fill out the following table.

Eigenvalue λ	Alg. Multiplicity of λ	Geo. Multiplicity of λ	A Basis for E_λ

(b) (3 points) What is a basis for range(A)?

(c) (3 points) Let $x = u_1 + u_2 + u_3$. What is $[A^{18}x]_{\mathcal{B}}$? You are allowed to have exponents in your answer.

6. Let A, u_1, u_2, u_3, u_4 be as defined in Question 5.
   (a) (2 points) What is $\det(A)$?

   (b) (2 points) What is $\det(A + 2I)$?

   (c) (2 points) What is $\operatorname{rank}(A)$?

   (d) (2 points) What is $\operatorname{rank}(A - 2I)$?

   (e) (2 points) Does $Ax = -u_2 + u_3 - 4u_4$ have a solution? If so, express it as a linear combination of u_1, u_2, u_3, u_4.

   (f) (2 points) Does $Ax = 2u_1 + u_2 + u_3$ have a solution? If so, express it as a linear combination of u_1, u_2, u_3, u_4.